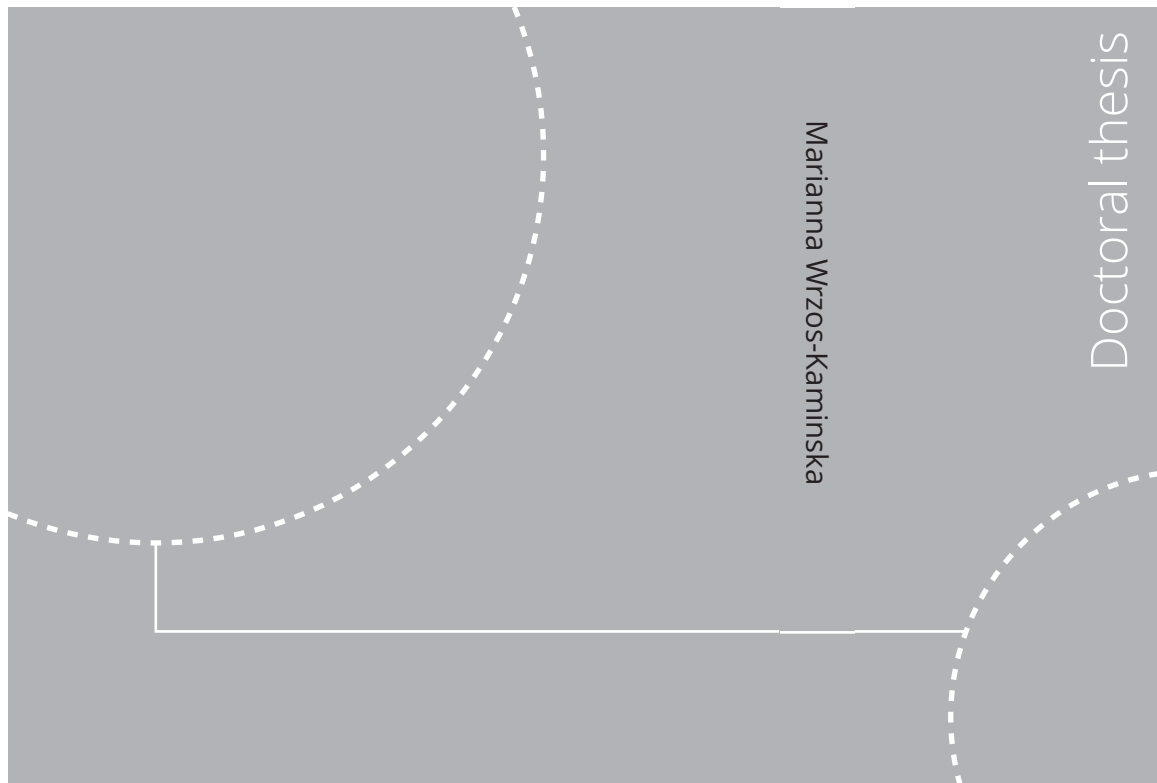


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Thesis for the degree of
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Faculty of Information Technology
and Electrical Engineering
Department of Engineering Cybernetics

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Thesis for the degree of Philosophiae Doctor

Trondheim, December 2025

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Summary

Robots are capable of performing tasks in environments which are dangerous or inaccessible to humans, such as underwater in deep waters. As an alternative to more traditionally shaped underwater vehicle-manipulator systems (UVMSs), which consist of a relatively large vehicle with a small arm attached, articulated intervention autonomous underwater vehicles (AIAUVs) have an articulated body similar to snake robots or a free-floating arm, with thrusters attached to the links. This design gives rise to new possibilities through the increased maneuverability and accessibility, but also new challenges due to its relatively light weight, and multiple actuators and degrees of freedom (DOFs) to coordinate.

This thesis considers the development of control methods for AIAUVs and their precursors, underwater snake robots (USRs). The latter are not equipped with thrusters, and instead they travel by swimming like their biological counterparts. The thesis is divided into two parts, one addressing control methods for AIAUVs, and one considering control of USRs. This second part also includes stability theory on uniform practical asymptotic stability, a useful concept for the analysis of systems such as swimming USRs.

The first part of this thesis includes control methods both for using an AIAUV as a free-floating arm, or as a vehicle when it travels over greater distances. To address the first of these scenarios, we apply control methods designed for robot manipulators performing multiple tasks, prioritized in a hierarchy and with time-varying references for each task. These control methods rely on a coordinate change using dynamically consistent null-space projections in order to decouple the dynamics of the new task coordinates. A new, robust controller based on this control framework is proposed in this thesis. In addition, new stability analysis of existing control methods is presented, proving stronger stability properties than in previous literature, namely global exponential stability in absence of singularities in the workspace. The performance of the newly proposed robust controller is compared against two other closely related control methods in simulation. The new controller makes the simulated robot manipulator successfully complete the assigned tasks, also when subject to an external disturbance.

The coordinate change used in the aforementioned control method for tracking of multiple tasks is then shown to coincide with the generalized Jacobian matrix (GJM) for a certain choice of the highest-priority task. The GJM was designed for control of free-floating vehicle-manipulator systems (VMSs) in space, to take into account the motion of the overall system induced by moving the attached manipulator arm. Relating the GJM to the more general task coordinates enables us to unify the two approaches and expand existing control approaches for VMSs in space. In those

control approaches, the GJM is used to decouple the control of the center of gravity (CG) of the overall VMS from control of the end-effector of the manipulator, and such that the thrusters are only engaged to move the CG, while the joints are used to move the end-effector. In this thesis, these control methods are extended to tracking of time-varying task references, and to an arbitrary number and choice of lower-priority tasks. The control method is then applied to AIAUVs in order to use their thrusters only to reposition the vehicle, while the joints are used for other tasks which may require a higher precision. To achieve this, we propose a thrust allocation scheme for AIAUVs based on the use of a (1,2,3)-pseudoinverse. The resulting control method is shown to render the set in which all tasks are completed uniformly asymptotically stable (UAS), and is validated in simulations.

In contrast to assigning strict priorities to tasks, this thesis also explores the use of an optimal control method in which multiple objectives are combined into one cost function, with soft priorities given by weights. The objective is to control the joints of an AIAUV to desired setpoints while avoiding self-collision and joint limits. In order to also incorporate robustness against disturbances we apply mixed H_2/H_∞ control, resolved as a nonlinear differential game with the control input and a disturbance being the players. Before applying a method of constructing approximate optimal solutions to such nonlinear differential games to an AIAUV, we use a feedback linearizing input to simplify its joint dynamics into a double integrator. Then a $\bar{P}$ -matrix solution, which is necessary to construct the approximate optimal control input, is found in this thesis for the double integrator dynamics. The overall control input consisting of the feedback linearizing terms combined with the solution to the differential game is simulated on a model of an AIAUV. Simulation are performed of both a worst-case disturbance, i.e. if the disturbance were a conscious player playing optimally, and with a more realistic disturbance due to uncertainties in the parameters used for the feedback linearizing input. The joints successfully reach the desired setpoints while avoiding collision, given a suitable initialization of the controller.

The final chapter of the first part of this thesis considers the AIAUV as a vehicle rather than an arm or UVMS, and addresses the problem of path following for such a vehicle. The objective is to use the joints of the AIAUV for direction control by curving the articulated body of the vehicle, while the thrusters are used only for forward propulsion. First, a control law for tracking of reduced attitude or pointing direction for a general system equipped with only two control torques is presented and analysed. Then, this control law is used as inspiration for specifying reference joint angles for the joints of the AIAUV in order to curve the AIAUV towards the path. These are combined with a guidance law to guide the vehicle onto the path, and the proposed control method is validated in simulation.

The second part of this thesis considers the topics of stability theory on uniform practical asymptotic stability, and control of swimming USRs. The notion of practical asymptotic stability is useful in the analysis of systems whose state does not necessarily converge to an exact point in the state space, but rather to a neighbourhood which can be made arbitrarily small through the choice of system parameters. In this thesis, a new theorem for local uniform practical asymptotic stability of cascaded systems consisting of uniformly practically asymptotically stable (UPAS)

subsystems is presented.

Finally, this thesis presents a control method for the automatic alignment of a swimming USR with the wake formed by an ocean current behind a bluff body. Inspired by weather optimal position control (WOPC) for ships, which is designed to point a ship against wind and currents, we define references for the heading and velocity of a USR. These are combined with low-level controllers so that the joints of a USR achieve a swimming gait with the desired heading and forward velocity. The resulting cascaded system is analysed and shown to be UPAS. The analysis is also applicable to other underactuated vehicles equipped with appropriate low-level heading velocity controllers. The proposed control method is validated in high-fidelity simulations of the fluid-structure interaction between the USR and the surrounding fluid. Several scenarios are simulated, both with a stationary wake, and with the wake rotating. The USR successfully aligns itself with the wake at the desired distance to the bluff body, and in the case of a slowly rotating wake it successfully follows the wake and realigns itself with it.

Preface

This thesis is submitted in partial fulfilment of the requirements for the degree of philosophiae doctor (PhD) at the Norwegian University of Science and Technology (NTNU). This work was conducted at the Centre for Autonomous Marine Operations and Systems (AMOS), Department of Engineering Cybernetics, under the supervision of Professor Kristin Y. Pettersen and the co-supervision of Professor Jan Tommy Gravdahl.

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While working on my PhD I've been very fortunate to visit two excellent scholars. I would like to thank Associate Professor Thulasi Mylvaganam at Imperial College London for hosting me, and for lending me so much of your time explaining and discussing your work and the project. Visiting you at London Imperial College and getting to work with you was a wonderful experience. I would also like to thank Dr. Elena Panteley for hosting me at The Laboratory of Signals and Systems (L2S) at CentraleSupélec. Thank you for your time and all your valuable insights.

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Acronyms

AIAUV	articulated intervention autonomous underwater vehicle
AUV	autonomous underwater vehicle
CB	center of buoyancy
CG	center of gravity
CM	center of mass
DOF	degree of freedom
ES	exponentially stable
GES	globally exponentially stable
GJM	generalized Jacobian matrix
I-AUV	intervention autonomous underwater vehicle
LES	locally exponentially stable
LOS	line-of-sight
LUS	locally uniformly stable
ROV	remotely operated vehicle
UA	uniformly attractive
UAS	uniformly asymptotically stable
UGAS	uniformly globally asymptotically stable
UPAS	uniformly practically asymptotically stable
US	uniformly stable
USR	underwater snake robot
UVMS	underwater vehicle-manipulator system
VMS	vehicle-manipulator system
WOHC	weather optimal heading control
WOPC	weather optimal position control

Chapter 1

Introduction

1.1 Motivation

Robotic systems are increasingly taking over tasks in environments which are dangerous or inaccessible to humans. Deep oceans are such an environment, and the world's oceans remain largely unexplored. In order to learn about the oceans and access their resources, there is an increasing amount of tasks to be performed underwater for both research and industry purposes. Such tasks range from exploration of far waters or archaeological sites, to inspection, maintenance and repair of an increasing amount of subsea infrastructure, clean-up of hazardous materials and more. While tasks such as surveying and mapping are already successfully being performed by autonomous underwater vehicles (AUVs), operations requiring a vehicle to interact physically with its environment, such as maintenance and repair of infrastructure, are still largely performed by remotely operated vehicles (ROVs) [164]. Such vehicles require a human operator and a supporting surface vessel to which the ROV is connected, making operations very costly and their range limited. To address these drawbacks, there are ongoing efforts towards the development into intervention autonomous underwater vehicles (I-AUVs), which would be capable of performing such intervention operations autonomously without the need for an operator and support vessel [126, 145]. To fully remove the need for a surface vessel even for deployment, the idea of resident I-AUVs has been conceived. Such resident I-AUVs would be stationed at the ocean floor, for example near subsea infrastructure, and deployed from there [57, 96].

Typical ROVs and many I-AUVs are underwater vehicle-manipulator systems (UVMSs) consisting of a large vehicle body with one or more attached arms [146]. These vehicles are typically built to hold a stable position rather than for speed and range, they are costly to deploy due to their size and required equipment, and their large bodies have limited access to confined spaces. An alternative solution is provided by articulated intervention autonomous underwater vehicles (AIAUVs), illustrated in Figure 1.1. AIAUVs consist of several links connected by motorized joints forming a slender, articulated body, with thrusters attached to the links [96, 151]. An AIAUV can be straightened for more efficient long-distance travel, or used as a free-floating robotic arm, due to which they have also been referred to as underwater swimming manipulators (USMs). The articulated body has benefits also



Figure 1.1: Two Eelume AIAUVs. (Courtesy of Eelume)

when intervention capabilities are not necessary, such as easier access and maneuverability, which has led to the development of similar vehicles for inspection and surveying specifically [6, 89]. The slender, articulated vehicle body makes it easier for such vehicles to access narrow or intricate spaces, such as caves, wrecks or gaps in between infrastructure elements. The jointed body can also be used to provide independent positioning of a camera and light source, or multiple cameras for inspection, exploration or mapping of e.g. wrecks [40].

The articulated body of AIAUVs is inspired by underwater snake robots (USRs), which in turn are inspired by biological swimming snakes and eels [76]. Inspired by the efficient locomotion of fish, eels and swimming snakes, USRs locomote by using their joints to produce a swimming motion, called a gait. USRs have been researched as an alternative to more typical vehicles due to their high flexibility and maneuverability, especially in terms of access to confined spaces. Snake robots have even been proposed as a means of exploring liquid reservoirs below the surface of other planets due to their ability to access them through narrow vents [120].

USRs and AIAUVs have many degrees of freedom (DOFs) due to their many joints, and both have their distinct challenges in terms of control. USRs are unable to control their position directly, and must move by swimming. AIAUVs on the other hand have multiple thrusters, which can be distributed on multiple links. Compared to typical UVMSs, both tend to be light vehicles, which makes them more vulnerable to disturbances, for example from the environment, effects arising from their own motion, or due to uncertainties in the control design. The characteristics of AIAUVs and USRs give rise to both new possibilities in terms of fully exploiting their capabilities, and challenges to be addressed when designing control methods. For AIAUVs this includes coordination of thrusters and joints, control allocation, and since they are more likely to be used for tasks which require precision, dealing with disturbances is important for such tasks. Furthermore, in order to extend mission duration and the reach of both AIAUVs and USRs it is of interest to consider energy

efficiency, and pursue energy autonomy. Enabling underwater robots to recharge and recover energy from their surroundings and natural phenomena, such as waves, currents or wakes, would enable them to operate in larger areas without being limited by the need to return to a charging station or support vessel at the surface. In order to enable AIAUVs and USRs to do this, much research and development is still needed of both hardware and control methods which can facilitate such energy harvesting.

In this thesis, control methods which make use of the capabilities of AIAUVs and USRs or aim to address some of the challenges related to control of these systems are investigated. An AIAUV can be used either as free-floating manipulator arm, as a vehicle, or even swimming like a USR, and control methods pertaining to all three of these cases are considered in this thesis.

1.2 Background

Control of UVMSs presents various challenges, several of which are outlined in [4]. Such challenges include complex and uncertain dynamic models, particularly due to hydrodynamic effects, as well as kinematic redundancy, and strong coupling between the dynamics of vehicle and manipulator. As AIAUVs can be considered a special type of UVMS, these challenges are also highly relevant when designing control methods for AIAUVs. Some of these challenges apply to control of USRs as well, particularly the aspect of model complexity and uncertainty.

1.2.1 Control of AIAUVs

An AIAUV can be viewed both as a free-floating manipulator arm and a reconfigurable vehicle, and control methods for AIAUVs tend to target one of these use-cases specifically.

Control of AIAUVs as a manipulator arm

An AIAUV is subject to all of the challenges outlined in [4] for UVMSs, some of which are further exacerbated by the comparably light weight of the vehicle, and the number and placement of thrusters. An AIAUV typically has many thrusters, enough to be overactuated in pose, but they are likely to be mounted on multiple links, causing the thruster configuration to change depending on the joint angles. As a manipulator arm, an AIAUV may possess enough joints to be redundant with respect to its control objective, meaning that it has more DOFs than necessary to complete the objective. In addition, it is effectively a large arm with no designated vehicle base. If one of its links is viewed as a base, the base is very light compared to the arm, and will therefore experience more disturbances from the motion of the joints and thus the motion of the other links. Consequently, common control methods for vehicle-manipulator systems (VMSs) in which the base is moved and then the manipulator arm is controlled similarly to a fixed-base manipulator may not work as well for an AIAUV. In summary, the main challenges when it comes to control of AIAUVs are redundancy, which calls for coordination of the available DOFs and inputs, light weight, and, like for all UVMSs, uncertain dynamics.

When a system is redundant with respect to its objective, as an AIAUV is likely to be when used as a free-floating manipulator, multiple solutions which accomplish the objective exist. This redundancy must then be resolved. An attractive way of resolving such redundancy is to exploit the surplus DOFs to accomplish additional tasks or objectives, and several methods of redundancy resolution have been applied to AIAUVs. A popular solution for robotic manipulators and VMSs, in particular UVMSs, is to specify multiple, strictly prioritized tasks, find inputs corresponding to each task, and then combine them such that lower-priority tasks do not interfere with higher-priority ones [140]. For UVMSs, the predominant approach is to separate the control architecture into two levels: kinematic and dynamic control. Treating system velocities as inputs to accomplish the given tasks, velocity references are found through inverse kinematics for the joints, as well as the vehicle base in the case of VMSs. These velocities are then tracked using an underlying dynamic controller. To respect the priorities assigned to the different tasks, the joint velocities required to achieve each task are projected into the null-space of higher-priority tasks so as to not interfere [33, 140]. Applying this approach to AIAUVs was proposed in [148], and combined with sliding mode control as a robust control method for tracking the velocity references in [17, 19, 68], including experimental validation in [19]. Sliding mode control is designed to dominate disturbances and force the system state onto a submanifold of the state-space on which the thus constrained system dynamics are asymptotically stable, or, in other words, on which the state "slides" towards the origin. In [150] the avoidance of singular configurations of the manipulator is incorporated by activating a corresponding task when the manipulator is near such a configuration, and deactivating i.e. disregarding it otherwise.

Instead of using velocity references as intermediate input signals to be tracked, redundancy resolution can be accomplished at the dynamic level so that input torques are assigned directly. This has been considered for AIAUVs in [8, 45, 132]. The control method from [8] relies on feedback linearization, where all dynamics of the system must be known so that they can be cancelled by the control input. In [45, 132], methods for fixed base manipulators from [44, 54] which use so-called dynamically consistent projections are applied to AIAUVs. Dynamically consistent projections ensure that multiple prioritized tasks are accomplished in such a way that control inputs to lower-priority tasks do not induce accelerations in the higher-priority tasks. The control methods applied in [45, 132] only require partial cancellation of the dynamics, but still require knowledge of dynamic parameters, including those pertaining to hydrodynamic effects. In [45], first-order sliding mode control is used to make the control method robust against uncertain parameters. In all of the works [8, 17, 19, 45, 68, 132] the thrusters and joint torques, directly or indirectly via base and joint velocities respectively, are used together to accomplish all tasks. The allocation of control inputs is done for all the tasks combined, without making any distinctions or assignments of which tasks to accomplish using which type of actuators.

If the base of a VMS is light compared to the manipulator arm, motion of the arm will cause the base to move as well. Such effects were observed in [20] in experiments with an AIAUV, in which the pose of the backmost link of the AIAUV was controlled by the thrusters while a joint was used to move the frontmost link. Even when using a sliding mode control algorithm which is designed to be robust against disturbances,

the joint motion led to errors in the base pose, and larger and more variable thrust inputs were required to control the base while the joint was moving. For control approaches where the objectives are completed by assigning the motion of the base and then the motion of the joints relative to it, disturbances to the base position will propagate and have to be counteracted reactively. A control method addressing this kind of dynamic coupling was applied to AIAUVs in [2], where the inverse kinematics problem of finding reference velocities was solved using the generalized Jacobian matrix (GJM). The GJM was introduced for VMSs in space [153], and takes into consideration the motion of the base induced by moving the joints. When using the GJM, the resulting joint reference velocities are designed to not change the total momentum of the system. This way a task such as moving the end-effector is performed correctly, taking into account the motion of the base while the CG of the overall system stays in place. This is especially important in kinematic control, since the GJM incorporates knowledge of what is an effect of the dynamics of the system, more precisely its inertia. The approach from [2] shows promise in simulation, but considers only controlling the joints by setting their velocities directly, and thrusters are not used, leaving the simulated vehicle floating in one place.

As an alternative to methods using null-space projection to ensure that the priority of tasks is respected, like in the previously mentioned works, multiple tasks can be accomplished sequentially solving an optimization problem for each task, starting with the tasks with the highest priority. This approach was applied to AIAUVs by [9]. The solutions for subsequent lower-priority tasks are constrained so that they do not interfere with the solutions already found for the higher-priority tasks. Tasks which consist of satisfying an inequality, such as e.g. collision avoidance, are accomplished by using control barrier functions, which are used to restrict the admissible inputs to guarantee that the inequality will be satisfied.

Control of AIAUVs as a vehicle

When considering an AIAUV as a vehicle, such as when travelling between sites or for survey work, the joints make it possible to reconfigure its shape. In [20], the joints are then used to keep the vehicle in a predefined fixed shape, either straight or bent in a C-shape, or to move the front link from side to side as if performing an inspection. The thrusters are then used to track a time-varying reference pose of the backmost link, designated as the vehicle base. In contrast to the directly actuated joints of an AIAUV, a reconfigurable vehicle can also be constructed by using unactuated joints to link together several AUVs, as presented in [10]. Each of the AUVs is equipped with thrusters, enabling them to move relative to each other in order to reconfigure the overall shape of the vehicle, or together as a fully actuated vehicle. AIAUVs, on the other hand, typically have actuated joints which can be used to reconfigure the shape. In both of the works [10, 20], the configuration, or shape, of the vehicle, is a reference determined beforehand, and the position control is resolved as tracking a time-varying trajectory.

Path-following control, where the vehicle has to follow a path, but without specifying the exact time, has been considered in 2D for AIAUVs by [16, 91, 149], where the vehicle is controlled by holding a reference forward velocity and following a heading given by a line-of-sight (LOS) guidance law. The LOS guidance law

guides the vehicle onto a path by setting the heading reference such that the vehicle is aimed at a point further ahead along the path [49]. In [16] the AIAUV is kept straight, and thrusters are used for both forward propulsion and turning. The controller from [16] was combined with a weighted thrust allocation algorithm in [91], where the thrusters furthest away from the CG and therefore with a longer lever arm were used more actively for turning. Alternatively, an AIAUV can also use its joints to form a curve while turning, similarly to how swimming USRs can turn by curving the overall shape of their body as in e.g. [76]. Curving an AIAUV towards the path is compared against keeping the vehicle exclusively in a straight shape in [149]. When curving the vehicle, all joints are bent at the same angle chosen to be proportional to the heading error, like what was done for control of USRs by e.g. [76]. Even when the AIAUV is curved are still also used for turning, but simulation results suggest that curving the vehicle reduces the thruster effort necessary make it converge onto a straight path. Using the joints to curve the vehicle can also be used as the sole means of turning. An AIAUV possessing only tail thrusters which provide forward propulsion was used in [135] to experimentally demonstrate docking along a spiral path, using only its joints for direction control. Rather than assigning the same angle to all joints, like in [149], the joint angle is scaled linearly so that the joint angles increase towards the back, as if using the vehicle body like a tail for turning. This is beneficial to hold the camera, which was mounted at the front of the vehicle and used to navigate to the docking station, more steady. Turning motion alone was investigated experimentally in [90], where tight U-turns are accomplished by curving a 3-link vehicle into an approximation of a circle arc with the desired turning radius. Thrusters on the front link are used for forward propulsion, and the method is demonstrated to work in experiments at sufficiently low velocities. All of the works [16, 90, 91, 135, 149] consider only planar motion.

1.2.2 Control of UVMSs

Control methods for UVMSs can be designed considering the vehicle and manipulator together or separately. When taking a decoupled approach, the coordination of vehicle and manipulator must be done through planning of references for vehicle position and joint or end-effector position references. These references can then be followed by two separate controllers for the vehicle and the manipulator, as in e.g. [36, 162]. Controlling the vehicle and manipulator separately simplifies the control design and makes it easier to make use of off-the-shelf components with simple built-in control interfaces, but the modular control design makes it more difficult to address the interaction between dynamics of the vehicle and manipulator.

Arguably the most popular method within control of UVMSs is the use of task-priority inverse kinematics, in which the priority of tasks is enforced by the use of null-space projections [145]. This method can be used to find reference velocities for either the manipulator alone, or for both manipulator and vehicle together. This is the method previously mentioned as having been applied to AIAUVs in [17, 19, 148]. The approach can also be used to coordinate multiple, collaborating UVMSs [143]. In [144], several examples are provided of how to decompose more complex objectives, such as preparing to grasp an object while simultaneously avoiding obstacles

and maintaining the object in the field of view of a camera, into prioritized tasks to be passed to an inverse kinematics algorithm. Control of interaction forces can be performed within this framework by measuring the force and adjusting the position and velocity references accordingly, as demonstrated in [27]. Typically, the force is either controlled directly, or the position trajectory is modified to achieve a desired impedance behaviour. These two approaches were combined to make the interaction force converge to a desired value by designing a variable stiffness in [34]. As an alternative to methods based on null-space projections, sequentially solving an optimization problem for each task was applied to UVMSs in [141].

Dynamic controllers which track the reference velocities found through an inverse kinematics algorithm can be designed for the UVMS as a whole, or separately for the vehicle and manipulator. Several dynamic controllers can be found in [4], including a robust sliding mode controller and an adaptive controller where system parameters are estimated online in order to cancel out dynamics without prior knowledge of parameters. If the dynamic control of vehicle and manipulator is separated, the dynamic coupling between base and manipulator can only be handled by cancelling it out, or designing a controller which is robust against the resulting disturbance. In [152], the forces acting on a lightweight vehicle due to the motion of an attached manipulator are computed based on the dynamical model in order to cancel them by feedback. This was later combined with a robust sliding mode controller for the vehicle [26].

Instead of finding velocity references which fulfil a set of tasks, forces and torques can be assigned directly. When the system dynamics are rewritten into the coordinates of the tasks in order to find inputs, this is referred to as operational space control [80]. In [37], the aim is to control the position of the manipulator end-effector while achieving a desired impedance behaviour, meaning that the end-effector behaves like a spring-mass-damper-system in response to a force acting on it from the environment. The reference position, velocity and acceleration is adjusted based on the measured force and tracked by a control input with adaptive control parameters. In [22], operational space control with successive projections of each control force into the null-space of all previous tasks was applied to a humanoid UVMS with two arms. Hydrodynamic effects are computed using a simplified model in order to cancel them [21]. Constraints such as joint limit avoidance, self-collision avoidance and obstacle avoidance are handled by activating them as tasks at the highest priority when near collision. When such a task is active, other tasks are projected into its null-space, and an input for the avoidance task is generated by the use of artificial potential fields, designed to artificially push parts of UVMS away from e.g. collisions. Position and force control in the operational or task space of the end-effector was applied to a fixed-base underwater manipulator in [7]. The interaction force at the end-effector is estimated based on the power consumption of the joints, and sliding mode control is used to provide robustness against imperfect cancellation of uncertain dynamics. The proposed method was applied in experiments in a pool with promising results. A similar approach was applied to UVMSs by [38], where the force acting on the end-effector is measured instead of estimated, and what is called the sliding surface, which describes the system behaviour to be enforced by the controller, is designed to give a desired impedance behaviour. Experiments

are performed using a vehicle with an end-effector mounted rigidly directly on the vehicle rather than connected by an arm.

1.2.3 Control of redundant robotic manipulators

When a robot has more DOFs than required to complete a task, it is redundant with respect to that task. To utilize all available DOFs, and occupy them to avoid undefined behaviour, a well-established approach is the use of null-space projections to perform multiple tasks while enforcing a strict order of priority between them [80]. Projecting the input required to complete lower-priority tasks into the null-space of higher-priority tasks ensures that lower-priority tasks do not disturb the higher-priority tasks. These inputs can be reference joint velocities [33, 140], as is popular within control of UVMSs, joint accelerations [130], or joint torques [80].

In the operational space formulation [80], the manipulator dynamics are transformed into the coordinates in which the task is given, and inputs are first found as forces in the task coordinates before they are mapped to joint torques and projected into the null-space of higher-priority tasks. Using dynamically consistent null-space projections makes it possible to decouple the inertia and consequently dynamics of tasks, ensuring that lower-priority tasks also do not accelerate higher-priority ones [42–44, 54, 118]. In [43], the stability properties of operational space control of prioritized regulation tasks using dynamically consistent projections is analysed, and it is shown that all task errors are minimized with respect to a specified potential function. If all tasks are compatible, all will be completed, and if they are incompatible the errors will be as small as possible when constrained by the higher-priority tasks. The case of tracking time-varying reference trajectories for prioritized tasks was analysed in [44], where local asymptotic stability of all tracking errors was proved under the condition that the tasks are compatible. A control scheme for tracking time-varying references which admits incompatible tasks was presented in [54]. While the tasks may be incompatible at times, their reference trajectories must still be compatible, meaning that the references together constitute a representation of the same manipulator motion.

Many of the control methods using joint accelerations or torques rely on perfect knowledge of the system dynamics in order to cancel them out completely [8, 80, 130] or partially [43, 44, 54, 118]. The impact of errors in the dynamic parameters has been investigated by [39, 69]. Only fixed, land-based manipulators were considered in [39], where it was concluded that errors in the inertia matrix affect only the performance during the transient. In particular, even in the case when there are errors in the inertia matrix used to compute the null-space projections which enforce priority, steady-state errors in lower-priority tasks still do not impact the higher-priority task. This is consistent with the resulting null-space projections now being weighted null-space projections, but no longer dynamically consistent as they no longer match the true dynamics of the manipulator [42]. In [69], hydrodynamic damping is also included, and it is shown that errors in the inertia matrix introduce a coupling of the task dynamics and a disturbance from inputs to other tasks, both lower- and higher-priority ones.

When redundancy resolution is performed at the velocity level, robustness against

uncertain dynamics as well as disturbances can be achieved by using a robust dynamic controller to track the resulting reference, like for AIAUVs in [17, 19], or for robotic manipulators in [129] where adaptive control is used to track the joint velocity references. Adaptive control has also been combined with redundancy resolution at torque level. In [55], an adaptive version of the control law from [54] is presented, where unknown dynamic parameters are estimated while always ensuring positive definiteness of the estimated inertia matrix. Strict priority between tasks is preserved, but only after all parameter estimates have converged to the true parameter values. Sliding mode control in operational space has previously been mentioned in the context of UVMSs, but has also been applied to a robotic hand in [66]. Robustness against constant or slowly varying external forces was achieved by [71, 131] through use of observers to estimate the external forces or torques. The estimate is used to cancel out the external force to achieve exact completion of the tasks. This is combined with impedance control in the remaining available null-space, where the impact of external forces is not cancelled out. That way, if a force is applied, the DOFs not occupied with accomplishing the tasks will be used to yield to the force without compromising the completion of the tasks.

Instead of using null-space projections to ensure strict priority of tasks, optimization techniques have been used by e.g. [70, 128], where priority is enforced by solving optimization problems sequentially for each task starting with the highest priority. If only a soft prioritizing of the tasks is desired, this can be achieved by weighting multiple objectives in one objective function, as in e.g. [47]. Optimization-based techniques also allow for easy incorporation of constraints, either directly [128], or through control barrier function like in [9].

In contrast, when using null-space projections, constraints are typically formulated as so-called set-based or inequality tasks, where the objective is to ensure that a variable remains in a set of values or that an inequality is satisfied. These tasks are realized by activating and deactivating them within the task hierarchy [41, 100, 104, 142]. When dynamics are not considered, but only velocities as in [104], high-priority inequality tasks can be enforced by simply projecting the velocities, and this projection needs only to be activated if the velocity is directed such that the constraint may be violated. When tasks are activated and deactivated, the Jacobians and consequently the resulting reference velocities or control torques will be discontinuous. Approaches to avoiding this kind of discontinuities exist [41, 100, 142], but the priority, i.e. non-interference between tasks may be lost during the transition while activating and deactivating a task.

Collision avoidance is the kind of objective that can be resolved by activating and deactivating tasks within a hierarchy [41, 104]. A similar approach was presented much earlier in [98], where desired joint velocities of a manipulator arm are modified to avoid an obstacle, so long as it does not interfere with the main task. This resembles the handling of lower-priority inequality tasks by [104], while collision avoidance is instead considered one of the highest priority tasks when active in [104]. Another early approach to solving collision avoidance for robotic manipulators was presented in [79]. Here, the method of artificial potential fields is used, where the control input is designed to make the robot act as if it is repelled away from obstacles and attracted towards a desired goal. The use of artificial potential

fields was combined with activation and deactivation of tasks in [41] and applied to the problem of self-collision avoidance for a humanoid robot. When active, the collision avoidance task has the highest priority, and the corresponding control input for the task is found using artificial potential fields. In order to avoid discontinuities while activating or deactivating the tasks, the computation of the pseudoinverse of the Jacobian is augmented to smooth the transition. As an alternative to resolving collision avoidance reactively by the controller, planning methods for planning of joint trajectories which avoid collisions of the robot with itself and with known static obstacles also exist [11].

1.2.4 Control of swimming underwater snake robots

Like their biological counterparts and inspiration, USRs move forward by swimming, mimicking the undulatory gait of biological snakes and eels. This is done by following a sinusoidal reference signal for the joint angles, so that a wave-like motion being propagated along the body. The motion of swimming USRs can then be controlled by varying the parameters of the reference signal. Different types of gaits, resembling either snakes or eels, can be obtained by varying the amplitude of the signal for each joint [75]. Lateral undulation, like the motion of biological swimming snakes, is typically characterized by a reference with the same amplitude for all joints, while a more eel-like gait can be obtained by increasing the amplitude of the joint references towards the back. In [155], an optimization algorithm is used to search for gait parameters which give the lowest energy consumption for a given desired forward speed. Depending on whether the joints were simulated with or without the ability to recover energy, the optimal gaits were found to be eel-like when energy recovery was possible, or if energy recovery was not possible, the resulting gaits were even more fish-like, using mainly the backmost link like a tail.

Swimming USRs are made to turn by adding an offset to the joint reference, so that the wave motion of the body oscillates about a curve. This approach has been applied to control swimming fish-like robots as well, examples of which include [62, 163]. In [154], a simplified dynamic model for the position and orientation of a swimming fish robot was found by combining averaging with fitting to simulation data. In the resulting model, the derivative of the turning rate is found to be proportional with the offset added to the sinusoidal motion of the tail. For snake robots, it has been observed in simulations and experiments, by e.g. [102, 133], that an offset in the joint angle references causes the robot to turn. Through numerical analysis of simulation results of both land-based and underwater snake robots, it was shown in [77, 93] that using an offset for turning makes the snake robot successfully converge to a path. Based on such observations, a simplified, control-oriented model has been developed for land-based snake robots [94] and extended to USRs in [72], where a simple model for the turning dynamics is proposed. Path following for USRs can be achieved by using a guidance law to determine the offset added to the joint when turning. Using a LOS guidance law, a USR was made to swim onto a straight path in [76], and similar guidance laws have been applied for USRs also in cases where the path to be followed is curved [86, 92].

Control of USRs in the presence of environmental disturbances such as ocean

currents has been considered by e.g. [63, 77, 85–87]. In [77, 85], an integral LOS guidance law is used to make a USR follow a straight path in the presence of a constant ocean current, and in [86] an observer is used to estimate a constant current and take it into account when following curved paths. A guidance scheme based on vector fields was used in [63] to make a USR follow a sinusoidal path in the presence of constant ocean currents, assuming that both heading and course, i.e. direction of the total velocity, could be found from measurements in order to align the USR to achieve the correct course. An adaptive control scheme was used in [87] to estimate and counteract the impact of a constant ocean current on the dynamics of a USR, in order to control its velocity and orientation. In [113], a control strategy was developed for USRs swimming in time-varying currents, where the objective was to make the USR swim to a desired position and maintain this position by swimming against a current. The time-varying current as well as its derivative must be known in order to fully realize the control input.

In works concerning USRs which analyse the stability properties of the proposed control schemes, stability analysis is commonly performed on the simplified control-oriented model [63, 85, 113], or numerically for a given simulated USR [77] due to the complexity of the system. To further complicate analysis, for undulating snake robots, and especially in the presence of disturbances from time-varying periodic currents or waves, their position will typically not converge to an exact point or trajectory, but rather oscillate around it. One approach to analysing such motion is to use the method of averaging to analyse the average motion as in e.g. [73, 154]. Another suitable concept is that of practical asymptotic stability [30], where a system is considered practically asymptotically stable if its state converges to a small neighbourhood of the origin, and the neighbourhood can be made arbitrarily small by choice of system parameters, such as tunable control parameters. This theory was applied to analyse control of swimming USRs in [113], where the closed-loop system was proved to be uniformly practically asymptotically stable (UPAS). Similarly, in [105] the velocity of a land-based snake robot was shown to converge to a small neighbourhood around its reference, but without using the terminology and theory of practical stability.

1.3 Contributions and outline

This thesis is divided into two parts. Part I addresses the control of AIAUVs, and consists of Chapters 2 to 6. Part II considers motion control of USRs and related stability theory on uniform practical asymptotic stability, and consists of Chapters 7 and 8. Finally, concluding remarks are given in Chapter 9.

Part I: Control of Articulated Intervention-AUVs

Chapter 2: Preliminaries for Part I

This chapter introduces background material relevant to Part I. First, the mathematical model of AIAUVs and the simulation model which will be used in subsequent chapters are presented. Next, two relevant topics within stability theory are introduced, specifically analysis of interconnected systems, and stability properties of

nested subsets of the state-space. These are two strategies which enable one to modularize the stability analysis of a system, and which will be applied to analyse the stability of some of the control methods proposed in Part I. Finally, a family of tracking controllers for robotic manipulators and VMSs performing multiple prioritized tasks is presented. These control methods will be built upon in the first two chapters of Part I.

Chapter 3: Exponentially Stable Robust Hierarchical Task Tracking Control

In this chapter a new, robust controller is proposed for robotic manipulators of VMSs performing multiple tracking tasks, and new analysis of two existing, similar control methods is presented. The new controller is validated in simulations of a 7-DOF robotic arm, and compared to the two most closely related existing controllers: its continuous predecessor, and a first-order sliding mode controller.

The new controller proposed in this chapter is a robust extension of a controller from [132], and is shown to make the origin of the tracking error dynamics globally exponentially stable in the presence of bounded disturbances, provided there are no singularities. The two existing controllers which are analysed in this chapter are the above-mentioned controller from [132] and another from [54]. For these controllers, stronger stability properties than previously are proved here, namely global exponential stability of the origin of the dynamics of all tracking errors in the absence of singularities in the workspace. These controllers were previously shown to give respectively local uniform asymptotic stability and global uniform asymptotic stability of origin of the task error dynamics, in both cases in the absence of singularities.

Chapter 4: Unifying the Generalized Jacobian Matrix with Prioritized Task Hierarchies

In this chapter, a control framework motivated by the goal of reducing the effort needed to counteract disturbances due to the manipulator motion is proposed for free-floating VMSs, such as AIAUVs. The proposed control method extends existing work on VMSs in space [60], and aims to use thrusters only for positioning of the overall vehicle and its workspace, by controlling the position of and momentum about the vehicle's CG, while the joints are used for tasks requiring precision such as end-effector motion.

First, the generalized Jacobian matrix (GJM) is shown to coincide with a dynamically consistent coordinate transformation for prioritized tasks for a particular choice of tasks. This enables us to extend methods using the GJM, such as [60], to an arbitrary number of prioritized tasks and tasks requiring tracking of time-varying trajectories, and we show that this can be done without losing the beneficial division of actuator use in which thrusters are engaged only for repositioning the overall VMS. In order to achieve this also for VMSs equipped with thrusters on multiple links, such as AIAUVs, a thrust allocation algorithm based on using a (1,2,3)-pseudoinverse is proposed in this chapter.

To achieve tracking of time-varying task references, the tracking controller from [132] is applied. The tasks to which we apply this controller in this chapter

include controlling the rotational momentum of the VMS, leaving the attitude of the vehicle free. As a result, when all control objectives are completed, the state of the VMS will have converged to within a subset of the configuration space rather than a single point as in [132] and its precursor [44]. We prove that the set in which all control objectives are completed is uniformly asymptotically stable using new stability theory results from [99]. Previous similar control methods for VMSs in space [59–61], in which attitude of the vehicle is left free, consider only regulation to constant setpoints.

Finally, the performance of the proposed control method is validated in a simulation study.

Chapter 5: Collision Avoidance using Mixed H_2/H_∞ Control

In this chapter, mixed H_2/H_∞ control is applied to control the joints of an AIAUV to desired setpoints while avoiding joint limits and self-collisions, in a manner which is robust to uncertainty in the parameters of the system dynamics. The problem of mixed H_2/H_∞ control is solved as a differential game in which the control input and a disturbance are viewed as the players, using a method for constructing approximate optimal solutions, i.e. optimal control input and worst-case disturbance, for nonlinear systems from [107]. In order to apply this method to the joints of an AIAUV, in this chapter a feedback linearizing input is used to simplify the joint dynamics into a double integrator. While the double integrator dynamics are linear, the cost functions considered will be nonlinear, making the resulting control problem nonlinear. A solution to a set of matrix equations called an algebraic $\bar{P}$ solution, which is necessary to apply the method from [107], is found in this chapter for the double integrator dynamics.

The approximate optimal control input for double integrator dynamics is combined with the feedback linearizing controller and cost functions incorporating barrier functions for the objectives of avoiding self-collision and joint limits. The performance of the resulting control method is validated in simulations with both the theoretical worst-case disturbance, and a disturbance due to errors in the dynamic parameters used in the feedback linearizing input.

Chapter 6: Path Following using Geometric Control of Reduced Attitude

This chapter considers path-following control for AIAUVs following straight paths, using only their joints for turning onto the path.

As an intermediate first step, a controller for tracking the pointing direction, also called the reduced attitude, of a general rigid body controlled by two torques is proposed in this chapter. The proposed controller uses a unit vector representation of the reduced attitude, and is an adaptation of a tracking controller presented in [23]. There, the reduced attitude is controlled by assigning angular accelerations directly, and inertia of the system is not considered in the case of the tracking problem. The use of a vector representation enables the control method to be singularity-free, in the sense that there are no singularities which correspond to a specific attitude. Instead, the proposed controller is able to make the reduced attitude converge to its reference from almost any initial reduced attitude, except the antipodal point,

i.e. the reduced attitude pointing in the exact opposite direction of the reference at the initial time, provided that the initial angular velocity satisfies a given bound. The proposed control law is proved to make the tracking error dynamics uniformly stable at the origin, and to make them converge exponentially to zero provided that the body velocity about the unactuated body axis is bounded.

Then, the reduced attitude controller is used as inspiration to design joint references for the joints of an AIAUV in order to make it turn while the thrusters are used for forward propulsion. This is combined with a LOS guidance law to form a path following controller for AIAUVs, and the performance of the proposed path-following controller is validated in simulation. Unlike previous path-following control methods for AIAUV which only consider planar motion [16, 135, 149], the method proposed in this chapter considers motion in 3D.

Part II: Practical Asymptotic Stability and Control of Underwater Snake Robots

Chapter 7: Uniform Practical Asymptotic Stability

This chapter introduces background material on local uniform practical asymptotic stability, and presents a new theorem for uniform practical asymptotic stability of cascades consisting of uniformly practically asymptotically stable (UPAS) subsystems. A preliminary cascade theorem for local uniform practical asymptotic stability was presented in [113]. In this chapter, we present a new intermediate result which enables us to relax the conditions of the cascade theorem from [113], and we restate the theorem with fewer conditions. Specifically, we prove that solutions starting sufficiently near the origin are bounded as a consequence of the other conditions posed in the cascade theorem, which eliminates the need to require boundedness of solution as an explicit condition to apply the theorem.

The notion of practical asymptotic stability is well suited for the analysis of motion of swimming USRs subject to disturbances from the environment, and will be used later in Chapter 8.

Chapter 8: Automatic Alignment of Underwater Snake Robots Operating in Wakes of Bluff Bodies

This chapter considers control of USRs swimming in the presence of unknown currents. The goal is to position the USR at a desired distance from a bluff body and automatically align it with the wake forming downstream from the bluff body due to an ocean current.

A joint controller developed for land-based snake robots in [93] is applied to control the joints of the USR to follow a swimming gait. We propose a new, simplified model for the velocity and orientation of a USR following a sinusoidal gait in the presence of time-varying currents, and design controllers for the velocity and heading of the USR which track desired time-varying references. The resulting closed-loop system consisting of the velocity and heading controllers combined with the joint controller is shown to be UPAS.

The design of heading and velocity references is based on weather optimal position control (WOPC) for ships [48, 82]. In this control approach the position of a ship is controlled so that it remains on the radius of a circle and points inwards, so that the disturbance due to wind and currents is able to push it like a pendulum to the point where it points directly against the mean environmental disturbance. The mean disturbance is assumed to be constant or slowly varying. In [48], only fully actuated ships were considered, while in [82], where underactuated ships were considered, the stability of the system was not analysed. In this chapter, we apply the method to a swimming USR, for which only the heading and forward velocity can be controlled. In addition, unlike [48, 82], we consider time-varying currents. We show that the reference heading and velocity successfully make the position of the USR converge to a neighbourhood of the desired position downstream along the center-line behind a bluff body, in the presence of time-varying currents. The resulting closed-loop system, consisting of joint controller, heading and velocity controllers, and their references, is shown to be UPAS, meaning that the neighbourhood to which solutions converge can be made arbitrarily small by varying parameters of the system. The analysis in this chapter holds also for vehicles other than USRs, provided that they are equipped with lower-level controllers capable of tracking a desired heading and forward velocity.

The proposed control method is validated through high-fidelity simulations that capture the intricate interaction between the USR and the surrounding fluid. The performance of the control method is investigated both when the direction of the current, and consequently the wake, is constant, as well as when it rotates. In the latter case, the USR has to follow the rotating wake to realign itself downstream in it.

Unlike most previous works on control of swimming USRs, in this chapter we consider the presence of time-varying currents. Compared to [113], where time-varying currents were included, the dynamic controllers in this chapter are developed based on more realistic models of the USR dynamics, and no knowledge or measurements of the current are necessary for the control method presented in this chapter.

Chapter 9: Conclusions and Future Work

This chapter presents concluding remarks and suggestions for future research.

Publications

The work presented in this thesis has resulted in the following publications:

Journal articles

- [115] A. Orucevic, M. Wrzos-Kaminska, J. T. Gravdahl and K. Y. Pettersen, ‘Uniform practical asymptotic stability for position control of underwater snake robots,’ *IEEE Trans. Control Syst. Technol.*, vol. 32, no. 5, pp. 1631–1646, 2024.
- [116] A. Orucevic, M. Wrzos-Kaminska, M. E. B. Lysø, K. Y. Pettersen and J. T. Gravdahl, ‘Automatic alignment of underwater snake robots operating in wakes of bluff bodies,’ *Control Eng. Pract.*, vol. 147, 2024.

Peer-reviewed conference papers

- [157] M. Wrzos-Kaminska, K. Y. Pettersen and J. T. Gravdahl, ‘Path following control for articulated intervention-AUVs using geometric control of reduced attitude,’ *IFAC-PapersOnLine*, vol. 52, no. 16, pp. 192–197, 2019, Proc. 11th IFAC Symp. Nonlinear Control Syst.
- [158] M. Wrzos-Kaminska, T. Mylvaganam, K. Y. Pettersen and J. T. Gravdahl, ‘Collision avoidance using mixed H_2/H_∞ control for an articulated intervention-AUV,’ in *Proc. 2020 European Control Conf.*, St. Petersburg, Russia, May 2020.
- [159] M. Wrzos-Kaminska, B. K. Sæbø, K. Y. Pettersen and J. T. Gravdahl, ‘Unifying the generalized Jacobian matrix and prioritized task hierarchies, with application to free-floating VMSs,’ *IFAC-PapersOnLine*, vol. 58, no. 20, 2024, Proc. 15th IFAC Conf. Control Appl. Mar. Syst. Robot. Vehicles.
- [160] M. Wrzos-Kaminska, J. I. Dyrhaug, B. K. Sæbø, J. T. Gravdahl and K. Y. Pettersen, ‘Exponentially stable robust hierarchical tracking control for vehicle-manipulator systems,’ in *Proc. 2025 IEEE Conf. Control Technol. Appl.*, (to appear), San Diego, California, USA, Aug. 2025.

Publications not part of this thesis

- [64] I. Gushkov, A. Orucevic, K. Y. Pettersen, W. Yao, M. Wrzos-Kaminska and J. T. Gravdahl, ‘Vector field path following of time-varying sinusoidal paths for underwater snake robots,’ *IEEE Trans. Control Syst. Technol.*, 2025, (submitted).

Part I

**Control of Articulated
Intervention-AUVs**

Chapter 2

Preliminaries for Part I

This chapter introduces the mathematical and simulation models of an AIAUV which will be used in Part I, as well as some background material on stability analysis and control methods for robot manipulators. The vehicle model is presented in Section 2.1. Section 2.2 introduces two theorems which enable different approaches to dividing the stability analysis of more complex systems into smaller subproblems, and which will be used in two of the subsequent chapters of Part I. Finally, Section 2.3 presents a control approach for robot manipulators and VMSs performing multiple tasks, which will be built upon later in Chapters 3 and 4.

2.1 Vehicle model

This section presents the mathematical model of an AIAUV and the simulation model which will be used in the subsequent chapters of Part I.

2.1.1 Mathematical modelling

An AIAUV can be modelled as $n+1$ cylindrical links connected by n 1-DOF revolute joints, numbered from back to front. Each link has a reference frame attached, placed at the back of the link with x -axis pointing forward along the link, and z -axis up [151]. An additional frame called the end-effector frame, denoted by subscript e , is attached to the front end of the front link, with the same orientation as the link frame. The reference frames are illustrated in Figure 2.1. The dynamics of an AIAUV can be modelled as those of a UVMS, where the backmost link is chosen as the base of the vehicle. The base link and front link may also occasionally be referred to as tail and head, respectively.

The configuration of the vehicle consists of the position $\mathbf{p} \in \mathbb{R}^3$ of the base frame, the attitude of the base frame relative to an inertial world frame, which can be represented by a quaternion $\mathbf{q} \in \mathbb{S}^3$, and the joint angles $\boldsymbol{\theta} \in \mathbb{R}^n$. The pose of the base frame is denoted $\boldsymbol{\eta} = [\mathbf{p}^\top, \mathbf{q}^\top]^\top$. The linear and angular velocities of the base given in the base frame are denoted $\mathbf{v} \in \mathbb{R}^3$ and $\boldsymbol{\omega} \in \mathbb{R}^3$, respectively, and taken together they form the twist $\boldsymbol{\nu} = [\mathbf{v}^\top \boldsymbol{\omega}^\top]^\top$. The complete generalized velocity of the system includes the joint velocities as well, and is denoted $\boldsymbol{\zeta} = [\boldsymbol{\nu}^\top \dot{\boldsymbol{\theta}}^\top]^\top \in \mathbb{R}^{6+n}$.

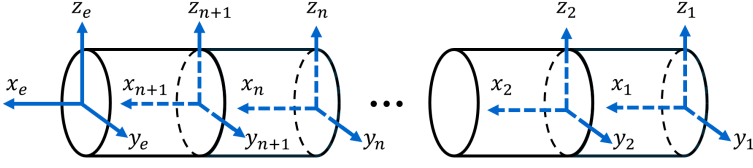


Figure 2.1: Illustration of the coordinate frames associated with each link.

The equations of motion are based on [51, 137] and are:

$$\dot{\eta} = T(q)\nu \quad (2.1a)$$

$$M(\theta)\dot{\zeta} + C(\theta, \zeta)\zeta + D(\theta, \zeta)\zeta + g(q, \theta) = B(\theta)u \quad (2.1b)$$

The transformation matrix $T(q)$ is

$$T(q) = \begin{bmatrix} R_b^I(q) & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{4 \times 3} & T_q(q) \end{bmatrix} \quad (2.2)$$

with

$$R_b^I(q) = I_3 + 2\eta S(\epsilon) + 2S(\epsilon)^2 \quad (2.3)$$

being the rotation matrix describing the orientation of the base frame, and

$$T_q(q) = \frac{1}{2} \begin{bmatrix} -\epsilon^\top \\ \eta I + S(\epsilon) \end{bmatrix} \quad (2.4)$$

where η , ϵ denote the scalar and vector part, respectively, of the quaternion q .

The matrix $M(\theta)$ in (2.1b) is the inertia matrix, $C(\theta, \zeta)$ is the Coriolis and centripetal matrix, $D(\theta, \zeta)$ is the hydrodynamic damping matrix and $g(q, \theta)$ is the vector of hydrostatic restoring forces. The actuator configuration matrix $B(\theta)$ describes the mapping from the input $u = [\tau_{\text{thr}}^\top, \tau_\theta^\top]^\top$ containing the thruster forces $\tau_{\text{thr}} \in \mathbb{R}^m$ from m thrusters placed on the links of the vehicle, and the joint torques $\tau_\theta \in \mathbb{R}^n$. The actuator configuration matrix can be written as

$$B(\theta) = \begin{bmatrix} B_{\text{thr}}(\theta) & \mathbf{0}_{6 \times n} \\ B_\theta(\theta) & I_n \end{bmatrix} \quad (2.5)$$

where $B_{\text{thr}}(\theta) \in \mathbb{R}^{6 \times m}$.

The dynamic model (2.1b) can be parametrized such that the inertia and damping matrices, $M(\theta)$ and $D(\theta, \zeta)\zeta$, are positive definite for all θ , ζ , and $\dot{M}(\theta) - 2C(\theta, \zeta)$ is skew-symmetric [4].

The inertia matrix $M(\theta)$ can be partitioned as

$$M(\theta) = \begin{bmatrix} M_b(\theta) & M_{bm}(\theta) \\ M_{bm}(\theta)^\top & M_m(\theta) \end{bmatrix} \quad (2.6)$$

where $M_b(\theta) \in \mathbb{R}^{6 \times 6}$, $M_{bm}(\theta) \in \mathbb{R}^{6 \times n}$, and $M_m(\theta) \in \mathbb{R}^{n \times n}$.



Figure 2.2: An Eelume AIAUV. (Courtesy of Eelume)

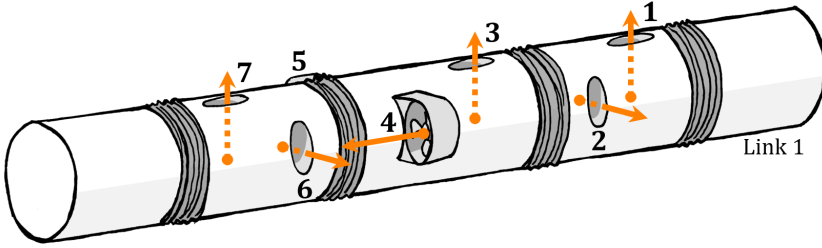


Figure 2.3: Illustration of the placement and direction of the thrusters on the simulated vehicle.

2.1.2 Simulation model

The simulation model is implemented in MATLAB/Simulink, based on [137]. The model is based on an AIAUV similar to that shown in Figure 2.2. This AIAUV has universal joints, which are modelled as two consecutive 1-DOF joints with a short link between them. The resulting model has $n = 8$ 1-DOF revolute joints, and 9 links. Joints 1, 3, 5, 7 rotate about the z -axis of the preceding link frame, and joints 2, 4, 6, 8 about the y -axis. The links have radius 0.085 m, and the individual link properties are given in Table 2.1. The vehicle has $m = 7$ thrusters. The notation "2: Z, Y" in Table 2.1 means that the link has two thrusters, acting in the z - and y -direction, respectively, in the link's frame. The simulated vehicle with its thrusters and their directions is illustrated in Figure 2.3. Each link is passively stabilized by ballast, which is simulated by placing the CG of each link below its center of buoyancy (CB), i.e. centroid, at a distance equal to half the radius.

Table 2.1: Link properties

Link no.	Length [m]	Mass [kg]	Thrusters
1	0.62	14.3	None
2, 4, 6, 8	0.104	6.0	None
3	0.584	12.7	2: Z, Y
5	0.726	9.8	3: Z, X, X
7	0.584	12.7	2: Y, Z
9	0.37	7.8	None

2.2 Stability of composite systems

This section presents background theory regarding two approaches to modularizing the stability analysis of dynamic systems which will be used later in this thesis. Section 2.2.1 presents an approach for systems which can be viewed as an interconnection of multiple subsystems, and their stability analysed by analysing the stability of each subsystems separately and then relating it to the magnitude of the interconnection terms.

Section 2.2.2 presents definitions and results regarding stability properties of (closed) sets. This allows the stability analysis to be modularized by analysing the stability properties of nested subsets of the state space relative to each other, i.e. analysing the stability properties of a set considering only states belonging to a larger subset of the state space.

2.2.1 Interconnected systems

The material in this section is from [78, Section 9.5], originally [5]. The theory introduced in this section will be applied in the analysis in Chapter 3.

Consider an interconnected system consisting of m subsystems of the form

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(t, \mathbf{x}_i) + \mathbf{g}_i(t, \mathbf{x}) \quad i = 1, \dots, m \quad (2.7)$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$, $\mathbf{x} = [\mathbf{x}_1^\top, \dots, \mathbf{x}_m^\top]^\top$ and $\sum_{i=1}^m n_i = n$. The functions $\mathbf{f}_i : \mathbb{R} \times \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_i}$, $\mathbf{g}_i : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}$ are assumed to be sufficiently smooth to ensure local existence and uniqueness of solutions, and to satisfy

$$\mathbf{f}_i(t, \mathbf{0}) = \mathbf{0}, \quad \mathbf{g}_i(t, \mathbf{0}) = \mathbf{0} \quad \forall i = 1, \dots, m \quad (2.8)$$

so that the origin is an equilibrium point of the system.

Suppose that for each subsystem there exists a positive definite, decrescent function $V_i : \mathbb{R} \times \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ satisfying

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial \mathbf{x}_i} \mathbf{f}_i(t, \mathbf{x}_i) \leq -\alpha_i \phi_i^2(\mathbf{x}_i) \quad (2.9a)$$

$$\left\| \frac{\partial V_i}{\partial \mathbf{x}_i} \right\| \leq \beta_i \phi_i(\mathbf{x}_i) \quad (2.9b)$$

for all $t \geq 0$ and $\|\mathbf{x}\| < r$ for some $r > 0$, where $\phi_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ are positive definite, continuous functions, and α_i, β_i are positive constants. Suppose further that the interconnection terms $\mathbf{g}_i(t, \mathbf{x})$ satisfy

$$\|\mathbf{g}_i(t, \mathbf{x})\| \leq \sum_{j=1}^m \gamma_{ij} \phi_j(\mathbf{x}_j) \quad (2.10)$$

for all $t \geq 0$ and $\|\mathbf{x}\| < r$, where γ_{ij} are non-negative constants.

A composite Lyapunov function candidate for the interconnected system is taken to be

$$V(t, \mathbf{x}) = \sum_{i=1}^m d_i V_i(t, \mathbf{x}_i) \quad (2.11)$$

where d_i are positive constants. Its derivative along trajectories of the interconnected system satisfies the inequality

$$\begin{aligned}\dot{V}(t, \mathbf{x}) &\leq \sum_{i=1}^m d_i \left(-\alpha_i \phi_i^2(\mathbf{x}_i) + \sum_{j=1}^m \beta_i \gamma_{ij} \phi_i(\mathbf{x}_i) \phi_j(\mathbf{x}_j) \right) \\ &= -\frac{1}{2} \boldsymbol{\phi}^\top (\mathbf{D}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}^\top \mathbf{D}) \boldsymbol{\phi}\end{aligned}\quad (2.12)$$

where $\boldsymbol{\phi} = [\phi_1(\mathbf{x}_1), \dots, \phi_m(\mathbf{x}_m)]^\top$, $\mathbf{D} = \text{diag}(d_1, \dots, d_m)$ and $\boldsymbol{\Sigma}$ is an $m \times m$ matrix with elements

$$\sigma_{ij} = \begin{cases} \alpha_i - \beta_i \gamma_{ii} & i = j \\ -\beta_i \gamma_{ij} & i \neq j \end{cases}. \quad (2.13)$$

Since all off-diagonal elements of $\boldsymbol{\Sigma}$ are non-positive, $\boldsymbol{\Sigma}$ is a *Z-matrix* [14, Definition 3.1.4].

Lemma 2.1 ([78], Lemma 9.7). *There exists a positive definite diagonal matrix $\mathbf{D}$ such that $\mathbf{D}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}^\top \mathbf{D}$ is positive definite if and only if $\boldsymbol{\Sigma}$ is a nonsingular M-matrix; that is, $\boldsymbol{\Sigma}$ is a Z-matrix and the leading principal minors of $\boldsymbol{\Sigma}$ are positive:*

$$\det \left(\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_{kk} \end{bmatrix} \right) > 0, \quad k = 1, \dots, m.$$

Remark. The term M-matrix has been used in the literature to denote Z-matrices whose eigenvalues all have either strictly positive real parts (e.g. in [5, 78]), or only nonnegative real parts (e.g. in [14]). In this thesis, the wording of Lemma 2.1 and Theorem 2.1 has been changed to use the term nonsingular M-matrix specifically.

Theorem 2.1 ([78], Theorem 9.2). *Consider the system (2.7) and suppose there are positive definite decrescent Lyapunov functions $V_i(t, \mathbf{x}_i)$ that satisfy (2.9a) and (2.9b) and that $\mathbf{g}_i(t, \mathbf{x})$ satisfies (2.10) for all $t \geq 0$ and $\|\mathbf{x}\| < r$. Suppose the matrix $\boldsymbol{\Sigma}$ defined by (2.13) is a nonsingular M-matrix. Then, the origin is uniformly asymptotically stable (UAS). Moreover, if all the assumptions hold globally and $V_i(t, \mathbf{x}_i)$ are radially unbounded, it will be uniformly globally asymptotically stable (UGAS).*

2.2.2 Reduction theorem for stability of compact sets for time-varying systems

The material in this section is from [99]. It will later be used in Chapter 4.

The systems considered are general system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \quad (2.14)$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies

Assumption 2.1 ([99], Basic assumption). The function $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is piecewise continuous with respect to its first argument and satisfies the following Lipschitz continuity property with respect to its second argument: for any compact set $K \subset \mathbb{R}^n$, there exists a constant $L > 0$ such that for each $\mathbf{x}_1, \mathbf{x}_2 \in K$ and for each $t \in \mathbb{R}$, $\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leq L\|\mathbf{x}_1 - \mathbf{x}_2\|$.

The right maximal interval of existence of a solution, i.e. the maximal interval contained in $\mathbb{R}_{\geq t_0}$ on which the solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ is defined, is denoted $T_{t_0, \mathbf{x}_0}^+$. For an interval $I \subset \mathbb{R}$ and subset $U \subset \mathbb{R}^n$, the following set is defined:

$$\mathbf{x}(I, t_0, U) = \{\mathbf{x}(t, t_0, \mathbf{x}_0) \in \mathbb{R}^n \mid t_0 \in I, \mathbf{x}_0 \in U\}. \quad (2.15)$$

So long as $I \subset T_{t_0, \mathbf{x}_0}^+$ for all $t_0, \mathbf{x}_0 \in \mathbb{R} \times U$, this set is well-defined.

Definitions

The following stability and attractivity properties are defined for compact sets:

Definition 2.1 ([99], Definition 3: Uniform stability and attractivity of compact sets). Consider (2.14) and let $\Gamma \subset \mathbb{R}^n$ be a compact set.

- Γ is uniformly stable (US) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathbf{x}(\mathbb{R}_{\geq t_0}, t_0, \mathcal{B}_\delta(\Gamma)) \subset \mathcal{B}_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$.
- Γ is uniformly globally stable (UGS) if Γ is US and for each $\delta > 0$ there exists $\varepsilon > 0$ such that $\mathbf{x}(\mathbb{R}_{\geq t_0}, t_0, \mathcal{B}_\delta(\Gamma)) \subset \mathcal{B}_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$.
- Γ is uniformly attractive (UA) if there exists $r > 0$ such that for each $\varepsilon > 0$ there exists $T > 0$ such that $\mathbf{x}(\mathbb{R}_{\geq t_0+T}, t_0, \mathcal{B}_r(\Gamma)) \subset \mathcal{B}_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$.
- Γ is uniformly globally attractive (UGA) if the UA property holds for all $r > 0$.
- Γ is uniformly asymptotically stable (UAS) if it is US and UA.
- Γ is uniformly globally asymptotically stable (UGAS) if it is UGS and UGA.

The properties in Definition 2.1 are analogous to the definitions of properties for equilibrium points in e.g. [78].

For closed sets, the following properties are defined:

Definition 2.2 ([99], Definition 5: t_0 -uniform stability and t_0 -uniform attractivity of closed sets). Consider system (2.14) and let $\Gamma \subset \mathbb{R}^n$ be a closed set.

- Γ is t_0 -uniformly stable (t_0 -US) if for each $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ such that $\Gamma \subset U$, and for each $x_0 \in U$, for each $t_0 \in \mathbb{R}$, and each $t \in T_{t_0, \mathbf{x}_0}^+$, it holds that $\mathbf{x}(t, t_0, \mathbf{x}_0) \in \mathcal{B}_\varepsilon(\Gamma)$.
- The *basin of t_0 -uniform attraction of Γ* is the set $\mathbb{B}(\Gamma)$ of initial states for which solutions converge to Γ uniformly with respect to t_0 :

$$\begin{aligned} \mathbb{B}(\Gamma) = \{ \mathbf{x}_0 \in \mathbb{R}^n \mid (\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R}) \\ t_0 + T \in T_{t_0, \mathbf{x}_0}^+ \text{ and } \mathbf{x}(\mathbb{R}_{\geq t_0+T} \cap T_{t_0, \mathbf{x}_0}^+, t_0, \mathbf{x}_0) \subset \mathcal{B}_\varepsilon(\Gamma) \}. \end{aligned}$$

- Γ is t_0 -uniformly attractive (t_0 -UA) if $\Gamma \subset \text{int}(\mathbb{B}(\Gamma))$.
- Γ is t_0 -uniformly globally attractive (t_0 -UGA) if $\mathbb{B}(\Gamma) = \mathbb{R}^n$.
- Γ is t_0 -uniformly asymptotically stable (t_0 -UAS) if Γ is t_0 -US and t_0 -UA.

- Γ is t_0 -uniformly globally asymptotically stable (t_0 -UGAS) if Γ is t_0 -US and t_0 -UGA.

The notions of t_0 -uniform stability and attractivity are properties which are uniform in initial time, but not initial state. For compact sets, several of the properties defined in Definition 2.1 and Definition 2.2 are equivalent; see [99, Proposition 9] for details.

Definition 2.3 ([99], Definition 8: Uniform boundedness of solutions). Let $\mathbf{x}_0 \in \mathbb{R}$. The solutions with initial state $\mathbf{x}_0$ are t_0 -uniformly bounded if there exist a constant $c > 0$ such that $\mathbf{x}(\mathbb{R}_{\geq t_0}, t_0, \mathbf{x}_0) \subset \mathcal{B}_c(\mathbf{0})$ for all $t_0 \in \mathbb{R}$. The set of initial states giving rise to t_0 -uniformly bounded solutions is defined as

$$\mathbb{BS} = \{\mathbf{x}_0 \in \mathbb{R}^n \mid (\exists c > 0)(\forall t_0 \in \mathbb{R})\mathbf{x}(\mathbb{R}_{\geq t_0}, t_0, \mathbf{x}_0) \subset \mathcal{B}_c(\mathbf{0})\}. \quad (2.16)$$

Several local properties of closed sets are also defined. These properties are local in the sense that they hold near a compact subset of the closed set:

Definition 2.4 ([99], Definition 10: Local uniform stability of Γ_2 near Γ_1). Let $\Gamma_1 \subset \Gamma_2$ be two closed subsets of $\mathbb{R}^n$, with Γ_1 compact. The set Γ_2 is *locally uniformly stable near Γ_1* (LUS- Γ_1) for (2.14) if there exists $r > 0$ such that for $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t_0 \in \mathbb{R}$ and any $\mathbf{x}_0 \in \mathcal{B}_\delta(\Gamma_1)$ the following implication holds:

$$(\forall t \in T_{t_0, \mathbf{x}_0}^+)(\mathbf{x}([t_0, t], t_0, \mathbf{x}_0) \subset \mathcal{B}_r(\Gamma_1) \Rightarrow \mathbf{x}([t_0, t], t_0, \mathbf{x}_0) \subset \mathcal{B}_\varepsilon(\Gamma_2)). \quad (2.17)$$

This is described as the set Γ_2 being LUS- Γ_1 if solutions starting sufficiently close to Γ_1 remain arbitrarily close to Γ_2 so long as they are contained in $\mathcal{B}_r(\Gamma_1)$. In other words, if Γ_2 is LUS- Γ_1 , solutions starting close to Γ_1 may move away from Γ_2 only after they have exited the neighbourhood $\mathcal{B}_r(\Gamma_1)$ first.

Definition 2.5 ([99], Definition 11: t_0 -uniform attractivity near a set). Consider system (2.14). The closed set $\Gamma_2 \subset \mathbb{R}^n$ is *t_0 -uniformly attractive near Γ_1* (t_0 -UA near Γ_1) if there exists $r > 0$ such that $\mathcal{B}_r(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$.

Finally, the following definitions are given for relative stability and attractivity properties:

Definition 2.6 ([99], Definition 12: Relative properties). Consider system (2.14), and let $\Gamma_1 \subset \Gamma_2$ be two closed subsets of $\mathbb{R}^n$, with Γ_1 compact.

- Γ_1 is US *relative to Γ_2* for (2.14) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathbf{x}(\mathbb{R}_{\geq t_0}, t_0, \mathcal{B}_\delta(\Gamma_1) \cap \Gamma_2) \subset \mathcal{B}_\varepsilon(\Gamma_1)$ for all $t_0 \in \mathbb{R}$.
- Γ_1 is UGS *relative to Γ_2* if Γ_1 is US relative to Γ_2 and for each $\delta > 0$ there exists $\varepsilon > 0$ such that $\mathbf{x}(\mathbb{R}_{\geq t_0}, t_0, \mathcal{B}_\delta(\Gamma_1) \cap \Gamma_2) \subset \mathcal{B}_\varepsilon(\Gamma_1)$.
- Γ_1 is UA *relative to Γ_2* if there exist $r > 0$ such that for each $\varepsilon > 0$ there exists $T > 0$ such that $\mathbf{x}(\mathbb{R}_{\geq t_0+T}, t_0, \mathcal{B}_r(\Gamma_1) \cap \Gamma_2) \subset \mathcal{B}_\varepsilon(\Gamma_1)$ for all $t_0 \in \mathbb{R}$.
- Γ_1 is UGA *relative to Γ_2* if $r > 0$ can be chosen arbitrarily large in the definition of UA relative to Γ_2 .
- Γ_1 is, respectively, UAS *relative to Γ_2* or UGAS *relative to Γ_2* , if Γ_1 is US (resp., UGS) and UA (resp., UGA) relative to Γ_2 .

Stability theorem and Lyapunov characterizations

Theorem 2.2 ([99], Theorem 18: Reduction theorem for uniform (global) asymptotic stability). *Consider the time-varying system (2.14) under Assumption 2.1. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Then Γ_1 is UAS if and only if*

- (i) Γ_1 is UAS relative to Γ_2 ,
- (ii) Γ_2 is LUS- Γ_1 , and
- (iii) Γ_2 is t_0 -UA near Γ_1 .

Moreover, Γ_1 is UGAS if and only if

- (i)' Γ_1 is UGAS relative to Γ_2 ,
- (ii) Γ_2 is LUS- Γ_1 , and
- (iii)' Γ_2 is t_0 -UGA, and
- (iv) all solutions are t_0 -uniformly bounded, i.e. $\mathbb{BS} = \mathbb{R}^n$.

Finally, if assumptions (i)', (ii) and (iii)' hold, then Γ_1 is UAS and all initial states giving rise to t_0 -uniformly bounded solutions are contained in the basin of t_0 -uniform attraction of Γ_1 , i.e., $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$.

If $\mathcal{X} \subset \mathbb{R}^n$ is a positively invariant set such that $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, and if the assumptions of Theorem 2.2 hold for initial states restricted to lie in $\mathcal{X}$, the the result of Theorem 2.2 holds relative to $\mathcal{X}$ [99, Remark 21].

Proposition 2.1 ([99], Proposition 25: Lyapunov characterization of the LUS- Γ_1 property). *Consider the time-varying system (2.14) under Assumption 2.1. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Suppose there exist $r, s > 0$ and C^1 non-negative function $V : \mathbb{R} \times \mathcal{B}_r(\Gamma_1) \rightarrow \mathbb{R}$ such that*

$$\alpha(\|\mathbf{x}\|_{\Gamma_2}) \leq V(t, \mathbf{x}) \leq \beta(\|\mathbf{x}\|_{\Gamma_1}) \quad (2.18)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}) \leq 0 \quad (2.19)$$

for all $(t, \mathbf{x}) \in \mathbb{R} \times \mathcal{B}_r(\Gamma_1)$, where $\alpha : [0, s) \rightarrow \mathbb{R}$ and $\beta : [0, r) \rightarrow \mathbb{R}$ are two class $\mathcal{K}$ functions. Then, Γ_2 is LUS- Γ_1 .

Corollary 2.1 ([99], Corollary 29: Lyapunov characterization of the property that Γ_2 is t_0 -UA near Γ_1). *Consider the time-varying system (2.14) under Assumption 2.1. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Suppose that Γ_1 is US, and for some open set $U \subset \mathbb{R}^n$ such that $\Gamma_1 \subset U$, there exists a C^1 non-negative function $V : \mathbb{R} \times U \rightarrow \mathbb{R}$ satisfying*

$$W_1(\mathbf{x}) \leq V(t, \mathbf{x}) \leq W_2(\mathbf{x}) \quad (2.20)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}) \leq -W_3(\mathbf{x}) \quad (2.21)$$

where $W_1, W_2, W_3 : U \rightarrow \mathbb{R}$ are continuous non-negative functions such that $W_1^{-1}(\mathbf{0}) = W_2^{-1}(\mathbf{0}) = W_3^{-1}(\mathbf{0}) = \Gamma_2 \cap U$. Then Γ_2 is t_0 -UA near Γ_1 .

2.3 Control methods for robot manipulators tracking multiple tasks

This section introduces background material relevant for Chapters 3 and 4. The material is mainly from [44, 132], with additional material from [45, 54, 161].

The control methods in this section are intended for robot manipulators or VMSs with many DOFs, which are to be utilized to accomplish multiple time-varying tasks. These tasks have specified priorities, forming a hierarchy of tasks. The control methods in this section are based on the use of dynamically consistent null-space projections to ensure that lower priority tasks do not interfere with higher-priority ones. The null-space projections are used in a coordinate change into new task coordinates with decoupled dynamics.

2.3.1 Dynamically consistent task coordinates

The method from [44] was originally developed for fixed-base manipulators, and applied to VMSs with dynamics on the form (2.1b) in [132]. The systems considered in [132] are actuated directly by a generalized force $\boldsymbol{\tau} \in \mathbb{R}^{6+n}$. This corresponds to setting $\mathbf{B}(\boldsymbol{\theta})\mathbf{u} = \boldsymbol{\tau}$ in (2.1b) and treating $\boldsymbol{\tau}$ as the control input.

Task variables

A task hierarchy comprised of $N \in \mathbb{N}$ tasks is considered, where each task $i = 1, \dots, N$ is defined in task-space coordinates as

$$\mathbf{x}_i = \mathbf{f}_i(\boldsymbol{\eta}, \boldsymbol{\theta}) \quad (2.22)$$

of dimension m_i . All the tasks together may also be collectively referred to as a stack of tasks. The tasks are subject to the following assumption [44]:

Assumption 2.2. The tasks are simultaneously feasible and the total dimension of the tasks is equal to the number of DOFs of the system, i.e. $\sum_{i=1}^N m_i = 6 + n$.

The task-space velocities are given by

$$\dot{\mathbf{x}}_i = \mathbf{J}_i(\boldsymbol{\eta}, \boldsymbol{\theta})\boldsymbol{\zeta} \quad (2.23)$$

where the task Jacobian matrices $\mathbf{J}_i(\boldsymbol{\eta}, \boldsymbol{\theta}) \in \mathbb{R}^{m_i \times (6+n)}$ are

$$\mathbf{J}_i(\boldsymbol{\eta}, \boldsymbol{\theta}) = \begin{bmatrix} \frac{\partial \mathbf{f}_i(\boldsymbol{\eta}, \boldsymbol{\theta})}{\partial \boldsymbol{\eta}} \mathbf{T}(\mathbf{q}) & \frac{\partial \mathbf{f}_i(\boldsymbol{\eta}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \end{bmatrix}. \quad (2.24)$$

The augmented Jacobian matrices $\mathbf{J}_i^{\text{aug}}(\boldsymbol{\eta}, \boldsymbol{\theta})$ are obtained by stacking Jacobian matrices for all tasks up to and including task i :

$$\mathbf{J}_i^{\text{aug}}(\boldsymbol{\eta}, \boldsymbol{\theta}) = \begin{bmatrix} \mathbf{J}_1 \\ \vdots \\ \mathbf{J}_i \end{bmatrix}. \quad (2.25)$$

It is necessary that the augmented Jacobian $\mathbf{J}_i^{\text{aug}}(\boldsymbol{\eta}, \boldsymbol{\theta})$ is invertible.

Assumption 2.3. The system operates within a subset of the state space which is free of singularities, and which contains the desired trajectory $\mathbf{x}_d(t)$ and an open neighbourhood around it.

In [44], where only fixed-base manipulators are considered, Assumption 2.3 is ensured by instead assuming that the entire considered workspace is free of singularities. For VMSs, the manipulator is only able to reach a subset of the entire possible workspace depending on the position of the base, and singularities may occur whenever the manipulator is at the boundary of its current reachable workspace, even though the point can be reached through a non-singular configuration from a different base position.

Augmented task-space velocities are defined as

$$\dot{\mathbf{x}}_i^{\text{aug}} = \mathbf{J}_i^{\text{aug}}(\boldsymbol{\eta}, \boldsymbol{\theta}) \boldsymbol{\zeta} \quad (2.26)$$

with $\dot{\mathbf{x}}_i^{\text{aug}} = [\dot{\mathbf{x}}_1^\top, \dots, \dot{\mathbf{x}}_i^\top]^\top$.

For each task, the objective is to track a desired, twice continuously differentiable time-varying trajectory $\mathbf{x}_{i,d}(t)$. The tracking errors are defined as

$$\tilde{\mathbf{x}}_i = \mathbf{x}_i - \mathbf{x}_{i,d}(t). \quad (2.27)$$

Augmented reference and error vectors are defined analogously to the augmented velocity vectors as

$$\mathbf{x}_{i,d}^{\text{aug}} = \begin{bmatrix} \dot{\mathbf{x}}_{1,d} \\ \vdots \\ \dot{\mathbf{x}}_{i,d} \end{bmatrix}, \quad \dot{\tilde{\mathbf{x}}}_i^{\text{aug}} = \begin{bmatrix} \dot{\tilde{\mathbf{x}}}_1 \\ \vdots \\ \dot{\tilde{\mathbf{x}}}_i \end{bmatrix}. \quad (2.28)$$

The augmented vectors for the last task, i.e. containing all the tasks, will be denoted

$$\dot{\mathbf{x}}_N^{\text{aug}} = \dot{\mathbf{x}}, \quad \mathbf{x}_{N,d}^{\text{aug}} = \mathbf{x}_d, \quad \dot{\tilde{\mathbf{x}}}_N^{\text{aug}} = \dot{\tilde{\mathbf{x}}}. \quad (2.29)$$

Dynamically consistent null-space projections and task velocities

To enforce the strict priority within the task hierarchy and avoid lower-level tasks interfering with ones of higher priority, new prioritized Jacobian matrices are found using dynamically consistent null-space projectors:

$$\bar{\mathbf{J}}_i(\boldsymbol{\eta}, \boldsymbol{\theta}) = \mathbf{J}_i(\boldsymbol{\eta}, \boldsymbol{\theta}) \mathbf{N}_i(\boldsymbol{\eta}, \boldsymbol{\theta})^\top \quad (2.30)$$

where the null-space projectors $\mathbf{N}_i(\boldsymbol{\eta}, \boldsymbol{\theta}) \in \mathbb{R}^{(6+n) \times (6+n)}$ can be found recursively as in [161]:

$$\mathbf{N}_i = \begin{cases} \mathbf{I}_{6+n} & \text{for } i = 1, \\ \mathbf{N}_{i-1} - \bar{\mathbf{J}}_{i-1}^\top \bar{\mathbf{J}}_{i-1}^{M+, \top} & \text{for } i = 2, \dots, N \end{cases} \quad (2.31)$$

where the state dependency on $\boldsymbol{\eta}, \boldsymbol{\theta}$ has been omitted for brevity. The operator $^{M+}$ denotes the dynamically consistent pseudoinverse, which is the weighted pseudoinverse with the mass matrix $\mathbf{M}$ as weight. For a matrix $\mathbf{A}$ with $n+6$ columns and full row rank, the dynamically consisted pseudoinverse can be found using the following expression:

$$\mathbf{A}^{M+} = \mathbf{M}^{-1} \mathbf{A}^\top (\mathbf{A} \mathbf{M}^{-1} \mathbf{A}^\top)^{-1}. \quad (2.32)$$

An expression which admits rank-deficient matrices is given in [54].

The prioritized Jacobians are stacked into

$$\bar{\mathbf{J}} = \begin{bmatrix} \bar{\mathbf{J}}_1 \\ \vdots \\ \bar{\mathbf{J}}_N \end{bmatrix}. \quad (2.33)$$

Using the prioritized Jacobians, dynamically consistent task-space velocities are defined as

$$\mathbf{v}_i = \bar{\mathbf{J}}_i \boldsymbol{\zeta} \quad (2.34)$$

with

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_N \end{bmatrix} = \bar{\mathbf{J}} \boldsymbol{\zeta} \quad (2.35)$$

Assumptions 2.2 and 2.3 ensure that $\bar{\mathbf{J}}$ is nonsingular, and $\bar{\mathbf{J}}^{-1}$ can then be expressed as [161]

$$\bar{\mathbf{J}}^{-1} = [\bar{\mathbf{J}}_1^{M+} \quad \dots \quad \bar{\mathbf{J}}_N^{M+}]. \quad (2.36)$$

Using (2.35), the VMS dynamics (2.1b) are transformed into task-space coordinates:

$$\underbrace{\bar{\mathbf{J}}^{-\top} \mathbf{M} \bar{\mathbf{J}}^{-1}}_{=\bar{\mathbf{M}}} \dot{\mathbf{v}} + \underbrace{\bar{\mathbf{J}}^{-\top} (\mathbf{C} - \mathbf{M} \bar{\mathbf{J}}^{-1} \dot{\bar{\mathbf{J}}}) \bar{\mathbf{J}}^{-1} \mathbf{v}}_{=\boldsymbol{\mu}} + \underbrace{\bar{\mathbf{J}}^{-\top} \mathbf{D} \bar{\mathbf{J}}^{-1} \mathbf{v}}_{=\boldsymbol{\delta}} = \bar{\mathbf{J}}^{-\top} (\boldsymbol{\tau} - \mathbf{g}). \quad (2.37)$$

Due to the use of dynamically consistent null-space projections, the transformed inertia matrix $\bar{\mathbf{M}}$ is block-diagonal with blocks $\bar{\mathbf{M}}_i = \bar{\mathbf{J}}_i^{M+\top} \mathbf{M} \bar{\mathbf{J}}_i^{M+}$. The transformed Coriolis and damping matrices $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ are in general not block-diagonal, and therefore introduce couplings between the task dynamics to be addressed when designing the control law.

The new, dynamically consistent task velocities are related to the original task-space velocities through

$$\mathbf{v} = \underbrace{\bar{\mathbf{J}}(\boldsymbol{\eta}, \boldsymbol{\theta}) \mathbf{J}_N^{\text{aug}}(\boldsymbol{\eta}, \boldsymbol{\theta})^{-1}}_{=\mathbf{G}(\boldsymbol{\eta}, \boldsymbol{\theta})} \dot{\mathbf{x}}_N^{\text{aug}}. \quad (2.38)$$

By Assumptions 2.2 and 2.3, $\mathbf{J}_N^{\text{aug}}$ is invertible in the considered subset of the state-space. The matrix $\mathbf{G}(\boldsymbol{\eta}, \boldsymbol{\theta})$ is lower triangular, and can be split into submatrices $\mathbf{G}_{i,j} \in \mathbb{R}^{m_i \times m_j}$ where $\mathbf{G}_{i,i} = \mathbf{I}_{m_i}$, and $\mathbf{G}_{i,j} = \mathbf{0}$ for $i < j$. This gives

$$\mathbf{v}_i = \dot{\mathbf{x}}_i + \sum_{j=1}^{i-1} \mathbf{G}_{i,j} \dot{\mathbf{x}}_j. \quad (2.39)$$

2.3.2 Tracking control

Several tracking controllers based on the coordinate transformation from Section 2.3.1 have been developed [44, 45, 54, 132, 161]. This section introduces two tracking controllers which use a modified velocity reference, and a robust version of one of them.

In [132], the modified velocity reference is formulated for original task coordinate velocities $\dot{\mathbf{x}}$, while in [45, 54] the modified reference velocity is for the dynamically consistent velocities $\mathbf{v}$.

Tracking using a modified velocity reference for original task velocities

The controller in [132] builds on a decoupling input introduced in [44] to cancel out remaining couplings between task dynamics, and introduces a new velocity reference.

Decoupling control input The dynamics (2.37) are no longer coupled through the inertia matrix, since it is block-diagonal, but they may still be coupled through the matrices $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$. In [44] a decoupling control input was introduced to cancel out remaining couplings, and [132] extended it with an extra term for cancelling the hydrodynamic damping effects:

$$\boldsymbol{\tau} = \mathbf{g} + \boldsymbol{\tau}_\mu + \boldsymbol{\tau}_\delta + \sum_{i=1}^N \mathbf{N}_i \mathbf{J}_i^\top \mathbf{F}_{i,\text{ctrl}} \quad (2.40)$$

where $\mathbf{F}_{i,\text{ctrl}}$ is a control input for task i . The terms $\boldsymbol{\tau}_\mu$, $\boldsymbol{\tau}_\delta$ cancel the contributions from off-diagonal blocks of the matrices $\boldsymbol{\mu}$, $\boldsymbol{\delta}$ in (2.37), and are given as:

$$\boldsymbol{\tau}_\mu = \sum_{i=1}^N \bar{\mathbf{J}}_i^\top \left(\sum_{j=1}^{i-1} \boldsymbol{\mu}_{i,j} \mathbf{v}_j + \sum_{j=i+1}^N \boldsymbol{\mu}_{i,j} \mathbf{v}_j \right), \quad (2.41a)$$

$$\boldsymbol{\tau}_\delta = \sum_{i=1}^N \bar{\mathbf{J}}_i^\top \left(\sum_{j=1}^{i-1} \boldsymbol{\delta}_{i,j} \mathbf{v}_j + \sum_{j=i+1}^N \boldsymbol{\delta}_{i,j} \mathbf{v}_j \right) \quad (2.41b)$$

where $\boldsymbol{\mu}_{i,j}$, $\boldsymbol{\delta}_{i,j} \in \mathbb{R}^{m_i \times m_j}$ denote blocks of the matrices $\boldsymbol{\mu}$, $\boldsymbol{\delta}$.

Inserting (2.40) into (2.37) fully decouples the dynamics of the dynamically consistent task-velocities, and the resulting dynamics of each task $i = 1, \dots, N$ are

$$\bar{\mathbf{M}}_i \dot{\mathbf{v}}_i + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \mathbf{v}_i = \mathbf{F}_{i,\text{ctrl}}. \quad (2.42)$$

For the primary task, it holds that $\mathbf{v}_1 = \dot{\mathbf{x}}_1$. For the lower-priority tasks with $i \geq 2$, the dynamics (2.42) can be transformed back into the original task coordinates by differentiating (2.39) and inserting for $\mathbf{v}_i$, $\dot{\mathbf{v}}_i$, resulting in

$$\bar{\mathbf{M}}_i \ddot{\mathbf{x}}_i + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \dot{\mathbf{x}}_i + \boldsymbol{\gamma}_i(\boldsymbol{\eta}, \boldsymbol{\theta}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\theta}}) \begin{bmatrix} \dot{\mathbf{x}}_{i-1}^{\text{aug}} \\ \ddot{\mathbf{x}}_{i-1}^{\text{aug}} \end{bmatrix} = \mathbf{F}_{i,\text{ctrl}} \quad (2.43)$$

where

$$\boldsymbol{\gamma}_i(\boldsymbol{\eta}, \boldsymbol{\theta}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\theta}}) = [\boldsymbol{\Gamma}_{i,1}, \dots, \boldsymbol{\Gamma}_{i,i-1}, \boldsymbol{\Psi}_{i,1}, \dots, \boldsymbol{\Psi}_{i,i-1}] \quad (2.44a)$$

$$\boldsymbol{\Gamma}_{i,j} = (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \mathbf{G}_{i,j} + \bar{\mathbf{M}}_i \dot{\mathbf{G}}_{i,j}, \quad \boldsymbol{\Psi}_{i,j} = \bar{\mathbf{M}}_i \mathbf{G}_{i,j}. \quad (2.44b)$$

The term including $\boldsymbol{\gamma}_i(\boldsymbol{\eta}, \boldsymbol{\theta}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\theta}})$ is a top-down disturbance from higher-priority tasks.

Task inputs In [132], a modified reference velocity $\dot{\mathbf{x}}_{i,r}$ is introduced for each task, chosen to be

$$\dot{\mathbf{x}}_{i,r} = \dot{\mathbf{x}}_{i,d} - \mathbf{\Lambda}_i \tilde{\mathbf{x}}_i \quad (2.45)$$

where the matrix $\mathbf{\Lambda}_i \in \mathbb{R}^{m_i \times m_i}$ is a positive definite gain matrix. A new error variable is defined as

$$\mathbf{z}_i = \dot{\mathbf{x}}_i - \dot{\mathbf{x}}_{i,r} = \dot{\mathbf{x}}_i + \mathbf{\Lambda}_i \tilde{\mathbf{x}}_i \quad (2.46)$$

and the control input $\mathbf{F}_{i,\text{ctrl}}$ to each task chosen as

$$\mathbf{F}_{i,\text{ctrl}} = \bar{\mathbf{M}}_i \ddot{\mathbf{x}}_{i,r} + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \dot{\mathbf{x}}_{i,r} - \mathbf{K}_{D,i} \mathbf{z}_i - \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i + \gamma_i \begin{bmatrix} \dot{\mathbf{x}}_{i-1,d}^{\text{aug}} \\ \ddot{\mathbf{x}}_{i-1,d}^{\text{aug}} \end{bmatrix} \quad (2.47)$$

where $\mathbf{K}_{P,i}, \mathbf{K}_{D,i} \in \mathbb{R}^{m_i \times m_i}$ are positive definite gain matrices.

The resulting closed-loop dynamics are given for each task i as

$$\dot{\tilde{\mathbf{x}}}_i = -\mathbf{\Lambda}_i \tilde{\mathbf{x}}_i + \mathbf{z}_i \quad (2.48a)$$

$$\bar{\mathbf{M}}_i \dot{\mathbf{z}}_i + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i} + \mathbf{K}_{D,i}) \mathbf{z}_i + \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i + \gamma_i \begin{bmatrix} \dot{\tilde{\mathbf{x}}}_{i-1}^{\text{aug}} \\ \ddot{\tilde{\mathbf{x}}}_{i-1}^{\text{aug}} \end{bmatrix} = \mathbf{0} \quad (2.48b)$$

The origin, in $\tilde{\mathbf{x}}$ - and $\mathbf{z}$ -coordinates, is shown to be locally UAS [132].

Tracking using a modified velocity reference for dynamically consistent task velocities

In [54], the task hierarchies are allowed to have a larger total dimension than the number of DOFs of the manipulator system executing the tasks. In this thesis, only scenarios in which Assumptions 2.2 and 2.3 hold will be considered. The controller from [54] was extended from fixed-based manipulators to UVMSs in [45], where robust versions of the controller were also presented.

In [54], a modified reference velocity $\mathbf{v}_r$ is introduced for the dynamically consistent task velocities. In the case where Assumptions 2.2 and 2.3 hold so that $\mathbf{J}_N^{\text{aug}}(\boldsymbol{\eta}, \boldsymbol{\theta})$ and $\bar{\mathbf{J}}(\boldsymbol{\eta}, \boldsymbol{\theta})$ are invertible, the modified reference velocity can be written as

$$\mathbf{v}_r = \mathbf{G}(\boldsymbol{\eta}, \boldsymbol{\theta}) \dot{\mathbf{x}}_d - \mathbf{\Lambda} \tilde{\mathbf{x}} \quad (2.49)$$

where $\mathbf{\Lambda}$ is a block diagonal matrix with positive-definite blocks $\mathbf{\Lambda}_i \in \mathbb{R}^{m_i \times m_i}$. The new error variable $\mathbf{s}$, also referred to as a sliding variable, is defined as

$$\mathbf{s} = \mathbf{v} - \mathbf{v}_r. \quad (2.50)$$

The control input is

$$\boldsymbol{\tau} = \mathbf{g} + \bar{\mathbf{J}}^\top (\bar{\mathbf{M}} \dot{\mathbf{v}}_r + (\boldsymbol{\mu}_d + \boldsymbol{\delta}_d) \mathbf{v}_r + (\boldsymbol{\mu}_s + \boldsymbol{\delta}_{od}) \mathbf{v} - \mathbf{K}_D \mathbf{s}) \quad (2.51)$$

where $\boldsymbol{\mu}_d$ and $\boldsymbol{\delta}_d$ denote the block-diagonal parts of $\boldsymbol{\mu}$, $\boldsymbol{\delta}$ respectively, $\boldsymbol{\mu}_s = \boldsymbol{\mu} - \boldsymbol{\mu}_d$ and $\boldsymbol{\delta}_{od} = \boldsymbol{\delta} - \boldsymbol{\delta}_d$, the former of which is skew-symmetric, and $\mathbf{K}_D$ is a block-diagonal gain matrix with positive-definite blocks $\mathbf{K}_{D,i} \in \mathbb{R}^{m_i \times m_i}$. The resulting closed-loop dynamics are then

$$\dot{\tilde{\mathbf{x}}} = -\mathbf{G}^{-1} \mathbf{\Lambda} \tilde{\mathbf{x}} + \mathbf{G}^{-1} \mathbf{s} \quad (2.52a)$$

$$\bar{\mathbf{M}} \dot{\mathbf{s}} = -(\boldsymbol{\mu}_d + \boldsymbol{\delta}_d + \mathbf{K}_D) \mathbf{s}. \quad (2.52b)$$

In the case where no singularities exist, i.e. if $\bar{\mathbf{J}}$ is always nonsingular, the origin is proved to be UGAS [45, 54].

In the cases where the total dimension of the task variables exceed the available DOFs, the gain matrices $\mathbf{\Lambda}$ and $\mathbf{K}_D$ introduced in the control input must be pre- and post-multiplied by $\bar{\mathbf{J}}\bar{\mathbf{J}}^{M+}$; see [45, 54] for details.

First-order sliding mode control In [45], a robust, first-order sliding mode version of the controller (2.51) is also introduced:

$$\boldsymbol{\tau} = \mathbf{g} + \bar{\mathbf{J}}^\top \left(\bar{\mathbf{M}}\dot{\mathbf{v}}_r + (\boldsymbol{\mu}_d + \boldsymbol{\delta}_d) \mathbf{v}_r + (\boldsymbol{\mu}_s + \boldsymbol{\delta}_{od}) \mathbf{v} - \mathbf{K}_D \mathbf{s} - \mathbf{K}_s \text{sgn}(\mathbf{s}) \right) \quad (2.53)$$

where $\text{sgn}(\cdot)$ is an element-wise sign function, and $\mathbf{K}_s \in \mathbb{R}^{6+n \times 6+n}$ is a positive definite diagonal matrix. Like $\mathbf{\Lambda}$ and $\mathbf{K}_D$, in the case of task dimensions exceeding the number of available DOFs, $\mathbf{K}_s$ must also be pre- and post-multiplied by $\bar{\mathbf{J}}\bar{\mathbf{J}}^{M+}$.

Chapter 3

Exponentially Stable Robust Hierarchical Task Tracking Control

This chapter presents a new, robust control law for robotic manipulators of VMSs performing multiple tracking tasks. By writing the resulting closed-loop system as an interconnected system, the new controller is shown to make all tracking errors globally exponentially stable in the presence of bounded disturbances, provided there are no singularities. This approach is also applied to two existing controllers, proving stronger stability properties than shown in previous literature for both controllers. The newly proposed control law is then validated in simulations of a robotic arm, and compared with two existing controllers.

Contributions of this chapter

The first contribution of this chapter is proving stronger stability properties than previously shown, namely global exponential stability of all tracking errors in the absence of singularities in the workspace, for the controllers proposed in [132] and [54]. These controllers were previously shown to give respectively local uniform asymptotic stability and global uniform asymptotic stability of the origin of the closed-loop system comprising all task tracking errors, in the absence of singularities. To enable this new analysis, the closed-loop system from [132] is rewritten in a way that makes it easier to compare with the controller and resulting closed-loop system from [54]. This rewriting also leads up to the second contribution of this chapter, namely proposing a new, robust version of the controller from [132]. This new robust controller is proved to give exponential stability of all tracking errors, globally if there are no singularities, in the presence of bounded disturbances if the gains are chosen sufficiently large. A robust controller was also proposed in [132], but stability and convergence of all tasks could not be proved. Finally, the performance of the controller proposed in this chapter is validated in simulations both in nominal conditions and under the influence of an external disturbance, and compared with the performance of two other controllers.

Publications

This chapter is based on [160].

Organization of this chapter

First, Section 3.1 gives a brief introduction. Section 3.2 presents the new robust controller and provides a stability analysis, including a stronger result for a previously proposed controller. Similarities and differences of the new control law in comparison with existing ones are first discussed in Section 3.3, before simulations investigating the performance of the new controller and comparing it against the performance of two of the other controllers are presented in Section 3.4. Then a summary of the chapter is given in Section 3.5. Finally, for completeness, new stability analysis of another of the control laws discussed in comparison with the one proposed in this chapter is given in Section 3.A.

3.1 Introduction

In [44], a tracking control law for robot manipulators tracking time-varying references for multiple, prioritized tasks was presented. This control law was then modified in [54] and [132], both of which propose to construct a new reference velocity. This new reference is constructed by adding a term ensuring convergence of the tracking error, similarly as one would when performing backstepping or designing a sliding variable. In [54], the new reference is designed for the dynamically consistent task velocities, which are in a sense relative velocities, while in [132], the new reference is designed for absolute task velocities. However, while the controller from [54] was proved to give uniform global asymptotic stability of the tracking errors in the absence of singularities, and was successfully extended into a robust first-order sliding mode controller in [45], in [132] only local stability properties could be shown. In addition, while a more robust version of the controller was also proposed in [132], it lacked a formal proof of stability of the perturbed closed-loop system, leaving the completion of tasks unverified.

In this chapter, a new robust version of the controller from [132] is proposed, for which global exponential stability of all task errors is proved, in the absence of singularities.

3.2 New proposed control law and stability analysis

This section presents the new proposed robust control law, the stability analysis of the resulting closed-loop system, and a new, stronger stability result for the controller [132], given in Section 2.3.2 by (2.40) and (2.47).

First, in order to facilitate and motivate the new control input, in Section 3.2.1 the closed-loop system resulting from applying the controller given by (2.40) and (2.47), is rewritten using the coordinate choice from [54] also introduced in Section 2.3.2.

This rearranging paves the way for the main theoretical contributions in this chapter. Namely, it enables the addition of an extra term to the controller from [132] in Section 3.2.2 to make it robust against bounded disturbances. The form of the closed-loop system dynamics makes it possible to show that the closed-loop system consisting of the entire stack of tasks is globally exponentially stable (GES) at the origin, in the absence of singularities in the workspace. This stability analysis, given in Section 3.2.3, can also be applied to the controller (2.40), (2.47) from [132], establishing stronger stability properties than previously, namely exponential stability of the tracking errors, and global exponential stability in the absence of singularities.

3.2.1 Rewriting the system dynamics into different coordinates

To motivate the new, robust control law which will be proposed, we first revisit the compliant control law (2.47) from [132], and the resulting closed-loop task dynamics. By relating the velocity errors $\mathbf{z}_i$ from [132] and given in (2.46), to the velocity error $\mathbf{s}$ introduced in [54] and given in (2.50), we can rewrite the closed-loop system (2.48) into a form which does not contain top-down disturbances from other task accelerations, and which will facilitate a different approach to analysing the stability of the closed-loop system.

The interconnection term which contains the top-down disturbance from other tasks in the closed-loop system (2.48) can be expressed as

$$\gamma_i(\boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\zeta}) \begin{bmatrix} \dot{\tilde{\mathbf{x}}}_{i-1}^{\text{aug}} \\ \ddot{\tilde{\mathbf{x}}}_{i-1}^{\text{aug}} \end{bmatrix} = \sum_{j=1}^{i-1} \left(\bar{\mathbf{M}}_i (\dot{\mathbf{G}}_{i,j} \dot{\tilde{\mathbf{x}}}_j + \mathbf{G}_{i,j} \ddot{\tilde{\mathbf{x}}}_j) + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \mathbf{G}_{i,j} \dot{\tilde{\mathbf{x}}}_j \right). \quad (3.1)$$

Using the expression (3.1), we can rewrite the closed loop dynamics (2.48b) as

$$\begin{aligned} & \bar{\mathbf{M}}_i \left(\dot{\mathbf{z}}_i + \sum_{j=1}^{i-1} \left(\dot{\mathbf{G}}_{i,j} \dot{\tilde{\mathbf{x}}}_j + \mathbf{G}_{i,j} \ddot{\tilde{\mathbf{x}}}_j \right) \right) \\ & + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \left(\mathbf{z}_i + \sum_{j=1}^{i-1} \mathbf{G}_{i,j} \dot{\tilde{\mathbf{x}}}_j \right) + \mathbf{K}_{D,i} \mathbf{z}_i + \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i = \mathbf{0}. \end{aligned} \quad (3.2)$$

Note that

$$\begin{aligned} \mathbf{z}_i + \sum_{j=1}^{i-1} \mathbf{G}_{i,j} \dot{\tilde{\mathbf{x}}}_j &= \boldsymbol{\Lambda}_i \tilde{\mathbf{x}}_i + \dot{\tilde{\mathbf{x}}}_i + \sum_{j=1}^{i-1} \mathbf{G}_{i,j} \dot{\tilde{\mathbf{x}}}_j \\ &= \underbrace{\dot{\tilde{\mathbf{x}}}_i + \sum_{j=1}^{i-1} \mathbf{G}_{i,j} \dot{\tilde{\mathbf{x}}}_j}_{=\mathbf{v}_i} - \left(\dot{\tilde{\mathbf{x}}}_{i,d} + \sum_{j=1}^{i-1} \mathbf{G}_{i,j} \dot{\tilde{\mathbf{x}}}_{j,d} - \boldsymbol{\Lambda}_i \tilde{\mathbf{x}}_i \right) \end{aligned} \quad (3.3)$$

where, since $\mathbf{G}$ is invertible by Assumptions 2.2 and 2.3, the expression in the last parenthesis is equal to the part of the modified reference velocity $\mathbf{v}_r$ defined in (2.49)

corresponding to the i th task. We therefore have

$$\mathbf{z}_i + \sum_{j=1}^{i-1} \mathbf{G}_{i,j} \dot{\tilde{\mathbf{x}}}_j = \mathbf{s}_i. \quad (3.4)$$

where $\mathbf{s}_i$ are the parts of $\mathbf{s} = [\mathbf{s}_1^\top, \dots, \mathbf{s}_N^\top]^\top$, the velocity error variable from [54], corresponding to each tasks. Inserting (3.4) into (2.48a) gives

$$\dot{\tilde{\mathbf{x}}}_i = -\mathbf{\Lambda}_i \tilde{\mathbf{x}}_i + \mathbf{s}_i - \sum_{j=1}^{i-1} \mathbf{G}_{i,j} \dot{\tilde{\mathbf{x}}}_j \quad (3.5)$$

which is equivalent to (2.52a) for the whole vector $\tilde{\mathbf{x}}$. Inserting (3.4) into (3.2) gives

$$\bar{\mathbf{M}}_i \dot{\mathbf{s}}_i + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \mathbf{s}_i + \mathbf{K}_{D,i} \left(\mathbf{s}_i - \sum_{j=1}^{i-1} \mathbf{G}_{i,j} \dot{\tilde{\mathbf{x}}}_j \right) + \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i = \mathbf{0}, \quad (3.6)$$

and consequently

$$\bar{\mathbf{M}} \dot{\mathbf{s}} + (\boldsymbol{\mu}_d + \boldsymbol{\delta}_d) \mathbf{s} + \mathbf{K}_D \mathbf{s} - \mathbf{K}_D (\mathbf{G} - \mathbf{I}) \dot{\tilde{\mathbf{x}}} + \mathbf{K}_P \tilde{\mathbf{x}} = \mathbf{0} \quad (3.7)$$

where $\boldsymbol{\mu}_d$, $\boldsymbol{\delta}_d$, $\mathbf{K}_D$ and $\mathbf{K}_P$ are block-diagonal matrices with blocks of dimensions corresponding to the task dimensions. Finally, inserting (2.52a) for $\dot{\tilde{\mathbf{x}}}$ in (3.7), the complete closed-loop dynamics are

$$\dot{\tilde{\mathbf{x}}} = -\mathbf{G}^{-1} \mathbf{\Lambda} \tilde{\mathbf{x}} + \mathbf{G}^{-1} \mathbf{s} \quad (3.8a)$$

$$\bar{\mathbf{M}} \dot{\mathbf{s}} + (\boldsymbol{\mu}_d + \boldsymbol{\delta}_d) \mathbf{s} + \mathbf{K}_D \mathbf{G}^{-1} \mathbf{s} + \mathbf{K}_D (\mathbf{I} - \mathbf{G}^{-1}) \mathbf{\Lambda} \tilde{\mathbf{x}} + \mathbf{K}_P \tilde{\mathbf{x}} = \mathbf{0}. \quad (3.8b)$$

In the case when $\bar{\mathbf{J}}$ is invertible, the inverse matrix $\mathbf{G}^{-1}$ is given by [54] as

$$\mathbf{G}^{-1} = \mathbf{J}^{\text{aug}} \bar{\mathbf{J}}^{-1} = \begin{bmatrix} \mathbf{I}_{m_1} & \mathbf{0}_{m_1 \times m_2} & \cdots & \mathbf{0}_{m_1 \times m_N} \\ \mathbf{J}_2 \bar{\mathbf{J}}_1^{M+} & \mathbf{I}_{m_2} & \cdots & \mathbf{0}_{m_2 \times m_N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{J}_N \bar{\mathbf{J}}_1^{M+} & \mathbf{J}_N \bar{\mathbf{J}}_2^{M+} & \cdots & \mathbf{I}_{m_N} \end{bmatrix}. \quad (3.9)$$

The system (3.8) can then be separated back into an interconnected system consisting of N subsystems, corresponding to each task, with subsystem number i for $i = 1, \dots, N$ given by

$$\dot{\tilde{\mathbf{x}}}_i = -\mathbf{\Lambda}_i \tilde{\mathbf{x}}_i + \mathbf{s}_i + \sum_{j=1}^{i-1} \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} (\mathbf{s}_j - \mathbf{\Lambda}_j \tilde{\mathbf{x}}_j) \quad (3.10a)$$

$$\begin{aligned} \bar{\mathbf{M}}_i \dot{\mathbf{s}}_i + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \mathbf{s}_i + \mathbf{K}_{D,i} \mathbf{s}_i \\ + \mathbf{K}_{D,i} \sum_{j=1}^{i-1} \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} (\mathbf{s}_j - \mathbf{\Lambda}_j \tilde{\mathbf{x}}_j) + \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i = \mathbf{0}. \end{aligned} \quad (3.10b)$$

Having rearranged the system dynamics into this form, we are ready propose the new, robust control input.

3.2.2 New robust control law

We will now address the scenario in which the system is subject to bounded disturbances, by augmenting the control input with an additional term to provide robustness against such bounded disturbances. Consider the system (2.37) with the decoupling control input (2.40), resulting in the closed-loop task dynamics (2.42). Suppose that the system is subject to a bounded disturbance, which can be transformed into task coordinates so that each task i is influenced by a bounded disturbance $\mathbf{f}_i(t, \mathbf{y}) \in \mathbb{R}^{m_i}$, and the task dynamics (2.42) become:

$$\bar{\mathbf{M}}_i \dot{\mathbf{v}}_i + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \mathbf{v}_i + \mathbf{f}_i(t, \mathbf{y}) = \mathbf{F}_{i,\text{ctrl}}. \quad (3.11)$$

Let the k th element of $\mathbf{f}_i(t, \mathbf{y})$ be denoted $f_{ik}(t, \mathbf{y})$.

Assumption 3.1. The disturbance is bounded, i.e. there exist positive constants L_{ik} such that

$$|f_{ik}(t, \mathbf{y})| \leq L_{ik} \quad (3.12)$$

for $k = 1, \dots, m_i$, $i = 1, \dots, N$.

To counteract the disturbance, we now propose to augment the control input (2.47) with an additional term designed to improve robustness of the controller:

$$\begin{aligned} \mathbf{F}_{i,\text{ctrl}} = & \bar{\mathbf{M}}_i \ddot{\mathbf{x}}_{i,r} + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \dot{\mathbf{x}}_{i,r} \\ & - \mathbf{K}_{D,i} \mathbf{z}_i - \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i - \mathbf{K}_{s,i} \text{sgn}(\mathbf{s}_i) + \gamma_i \begin{bmatrix} \dot{\mathbf{x}}_{i-1,d}^{\text{aug}} \\ \ddots \\ \dot{\mathbf{x}}_{i-1,d}^{\text{aug}} \end{bmatrix} \end{aligned} \quad (3.13)$$

where $\text{sgn}(\cdot)$ is the element-wise sign function and $\mathbf{K}_{s,i} \in \mathbb{R}^{m_i \times m_i}$ is a positive definite, diagonal gain matrix with diagonal elements $K_{s,ik}$, $k = 1, \dots, m_i$. The task dynamics (3.11) in closed loop with the input (3.13) and transformed into the error variables $\mathbf{s}_i$ become

$$\dot{\tilde{\mathbf{x}}}_i = -\boldsymbol{\Lambda}_i \tilde{\mathbf{x}}_i + \mathbf{s}_i + \sum_{j=1}^{i-1} \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} (\mathbf{s}_j - \boldsymbol{\Lambda}_j \tilde{\mathbf{x}}_j) \quad (3.14a)$$

$$\begin{aligned} \bar{\mathbf{M}}_i \dot{\mathbf{s}}_i + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \mathbf{s}_i + \mathbf{K}_{D,i} \mathbf{s}_i + \mathbf{K}_{D,i} \sum_{j=1}^{i-1} \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} (\mathbf{s}_j - \boldsymbol{\Lambda}_j \tilde{\mathbf{x}}_j) \\ + \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i + \mathbf{K}_{s,i} \text{sgn}(\mathbf{s}_i) + \mathbf{f}_i(t, \mathbf{y}) = \mathbf{0}. \end{aligned} \quad (3.14b)$$

3.2.3 Stability analysis

We are now ready to state the main result regarding the stability properties of the system in closed loop with the new proposed robust control law (3.13). This will be followed by a corollary which states stronger stability properties for the previous controller (2.47) from [132].

Theorem 3.1. *Consider the closed-loop system consisting of N subsystems with dynamics of subsystem i , $i = 1, \dots, N$ given by (3.14). Let Assumptions 2.2, 2.3 and 3.1 be satisfied, and suppose that the gain matrices $\mathbf{K}_{s,i}$ are chosen such that $K_{s,ik} \geq L_{ik}$ for all $k = 1, \dots, m_i$, $i = 1, \dots, N$. Then the origin is exponentially stable (ES), and if Assumption 2.3 holds globally, the origin is GES.*

Proof. For each subsystem (3.14), take the Lyapunov function candidate

$$V_i(\tilde{\mathbf{x}}_i, \mathbf{s}_i) = \frac{1}{2} \tilde{\mathbf{x}}_i^\top \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i + \frac{1}{2} \mathbf{s}_i^\top \bar{\mathbf{M}}_i \mathbf{s}_i. \quad (3.15)$$

Differentiating V_i gives

$$\begin{aligned} \dot{V}_i &= -\tilde{\mathbf{x}}_i^\top \mathbf{K}_{P,i} \Lambda_i \tilde{\mathbf{x}}_i + \tilde{\mathbf{x}}_i^\top \mathbf{K}_{P,i} \mathbf{s}_i + \tilde{\mathbf{x}}_i^\top \mathbf{K}_{P,i} \sum_{j=1}^{i-1} \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} (\mathbf{s}_j - \Lambda_j \tilde{\mathbf{x}}_j) \\ &\quad - \mathbf{s}_i^\top (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \mathbf{s}_i - \mathbf{s}_i^\top \mathbf{K}_{D,i} \mathbf{s}_i - \mathbf{s}_i^\top \mathbf{K}_{D,i} \sum_{j=1}^{i-1} \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} (\mathbf{s}_j - \Lambda_j \tilde{\mathbf{x}}_j) \\ &\quad - \mathbf{s}_i^\top \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i - \mathbf{s}_i^\top \mathbf{K}_{s,i} \text{sgn}(\mathbf{s}_i) - \mathbf{s}_i^\top \mathbf{f}_i(t, \mathbf{y}) + \frac{1}{2} \mathbf{s}_i^\top \dot{\bar{\mathbf{M}}}_i \mathbf{s}_i \\ &= -\tilde{\mathbf{x}}_i^\top \mathbf{K}_{P,i} \Lambda_i \tilde{\mathbf{x}}_i - \mathbf{s}_i^\top (\boldsymbol{\delta}_{i,i} + \mathbf{K}_{D,i}) \mathbf{s}_i - \mathbf{s}_i^\top \mathbf{K}_{s,i} \text{sgn}(\mathbf{s}_i) - \mathbf{s}_i^\top \mathbf{f}_i(t, \mathbf{y}) \\ &\quad + [\tilde{\mathbf{x}}_i^\top \ \mathbf{s}_i^\top] \begin{bmatrix} \mathbf{K}_{P,i} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{D,i} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{m_i} \\ -\mathbf{I}_{m_i} \end{bmatrix} \sum_{j=1}^{i-1} \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} \begin{bmatrix} -\mathbf{I}_{m_j} & \mathbf{I}_{m_j} \end{bmatrix} \begin{bmatrix} \Lambda_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m_j} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_j \\ \mathbf{s}_j \end{bmatrix}, \end{aligned} \quad (3.16)$$

using the fact that $\dot{\bar{\mathbf{M}}}_i - 2\boldsymbol{\mu}_{i,i}$ is skew-symmetric. The cross-terms due to the top-down disturbance from other tasks can be bounded by the product of the norms of each matrix and vector in the term, arriving at

$$\dot{V}_i \leq -\alpha_i \|\mathbf{y}_i\| + \sum_{j=1}^{i-1} \beta_{ij} \|\mathbf{y}_i\| \cdot \|\mathbf{y}_j\| - \sum_{k=1}^{m_i} |s_{ik}| (K_{s,ik} - |f_{ik}(t, \mathbf{y})|) \quad (3.17)$$

where $\mathbf{y}_i = [\tilde{\mathbf{x}}_i^\top \ \mathbf{s}_i^\top]^\top$, and

$$\alpha_i = \min(\lambda_{\min}(\mathbf{K}_{P,i} \Lambda_i), \lambda_{\min}(\boldsymbol{\delta}_{i,i} + \mathbf{K}_{D,i})), \quad (3.18a)$$

$$\begin{aligned} \beta_{ij} &= 2 \max(\lambda_{\max}(\mathbf{K}_{P,i}), \lambda_{\max}(\mathbf{K}_{D,i})) \cdot \sigma_{\max}(\mathbf{J}_i \bar{\mathbf{J}}_j^{M+}) \\ &\quad \cdot \max(\lambda_{\max}(\Lambda_j), 1). \end{aligned} \quad (3.18b)$$

Now let $\mathbf{y} = [\mathbf{y}_1^\top, \dots, \mathbf{y}_N^\top]^\top$, and let $V(\mathbf{y}) = \sum_{i=1}^N a_i V_i(\mathbf{y}_i)$ be a Lyapunov function candidate for the whole interconnected system, with $a_i, i = 1, \dots, N$ being positive constants. Then

$$\dot{V} \leq -\frac{1}{2} \boldsymbol{\phi}^\top (\mathbf{A} \boldsymbol{\Sigma} + \boldsymbol{\Sigma}^\top \mathbf{A}) \boldsymbol{\phi} - \sum_{i=1}^N \sum_{k=1}^{m_i} a_i |s_{ik}| (K_{s,ik} - |f_{ik}(t, \mathbf{y})|) \quad (3.19)$$

where $\boldsymbol{\phi} = [\|\mathbf{y}_1\|, \dots, \|\mathbf{y}_N\|]^\top$, $\mathbf{A} \in \mathbb{R}^{N \times N}$ is a diagonal matrix with the coefficients a_i along its diagonal, and the matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{N \times N}$ is given by

$$\boldsymbol{\Sigma} = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ -\beta_{21} & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{N1} & -\beta_{N2} & \cdots & \alpha_N \end{bmatrix}. \quad (3.20)$$

By Lemma 2.1, there exist constants a_i such that the matrix $\mathbf{A}\mathbf{\Sigma} + \mathbf{\Sigma}^\top \mathbf{A}$ is positive definite if and only if $\mathbf{\Sigma}$ is a nonsingular M-matrix, i.e. all its off-diagonal elements are non-positive and all leading principal minors are strictly positive. Since $\mathbf{\Sigma}$ is a lower-triangular matrix with strictly positive elements on the diagonal and non-positive elements below it, it is an M-matrix. Then by Theorem 2.1, the origin of the system (3.8) is UAS, and UGAS if Assumption 2.3 holds globally. At the origin, from (3.8a) it follows that $\dot{\tilde{\mathbf{x}}} = 0$, and consequently from (3.4) it follows that $\mathbf{z} = 0$, i.e. that the origin in $(\tilde{\mathbf{x}}, \mathbf{s})$ - and $(\tilde{\mathbf{x}}, \mathbf{z})$ -coordinates coincides. Thus the origin of (2.48) is UGAS if there are no singularities.

To prove exponential stability, we rewrite V as

$$V(\mathbf{y}) = \frac{1}{2} \mathbf{y}^\top \underbrace{\begin{bmatrix} a_1 \mathbf{K}_{P,1} & 0 & \cdots & 0 & 0 \\ 0 & a_1 \bar{\mathbf{M}}_1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & a_N \mathbf{K}_{P,N} & 0 \\ 0 & 0 & \cdots & 0 & a_N \bar{\mathbf{M}}_N \end{bmatrix}}_{=\mathbf{P}} \mathbf{y}, \quad (3.21)$$

and since $\mathbf{K}_{P,i}$, $\bar{\mathbf{M}}_i$ are positive definite matrices for all $i = 1, \dots, N$, we can choose $k_1 = \frac{1}{2} \lambda_{\min}(\mathbf{P}) > 0$ and $k_2 = \frac{1}{2} \lambda_{\max}(\mathbf{P})$ s.t.

$$k_1 \|\mathbf{y}\|^2 \leq V \leq k_2 \|\mathbf{y}\|^2. \quad (3.22)$$

For $K_{s,ik} \geq |f_{ik}(t, \mathbf{y})|$, it follows that

$$\dot{V} \leq -\frac{1}{2} \phi^\top (\mathbf{A}\mathbf{\Sigma} + \mathbf{\Sigma}^\top \mathbf{A}) \phi \leq -\frac{1}{2} \lambda_{\min}(\mathbf{A}\mathbf{\Sigma} + \mathbf{\Sigma}^\top \mathbf{A}) \|\phi\|^2 \quad (3.23)$$

where since $\mathbf{\Sigma}$ is a nonsingular M-matrix, by Lemma 2.1 there exists a choice of the matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{\Sigma} + \mathbf{\Sigma}^\top \mathbf{A}$ is a positive definite matrix and thus $\lambda_{\min}(\mathbf{A}\mathbf{\Sigma} + \mathbf{\Sigma}^\top \mathbf{A}) > 0$. Finally, by noting that

$$\|\phi\|^2 = \sum_{i=1}^N \|\mathbf{y}_i\|^2 = \sum_{i=1}^N \sum_{j=1}^{m_i} y_{i,j}^2 = \|\mathbf{y}\|^2 \quad (3.24)$$

where $y_{i,j}$ denotes the j th element of $\mathbf{y}_i$, we can bound $\dot{V}$ by

$$\dot{V} \leq -\frac{1}{2} \lambda_{\min}(\mathbf{A}\mathbf{\Sigma} + \mathbf{\Sigma}^\top \mathbf{A}) \|\mathbf{y}\|^2 \leq -\frac{\lambda_{\min}(\mathbf{A}\mathbf{\Sigma} + \mathbf{\Sigma}^\top \mathbf{A})}{\lambda_{\max}(\mathbf{P})} V = -k_3 V \quad (3.25)$$

with $k_3 = \frac{\lambda_{\min}(\mathbf{A}\mathbf{\Sigma} + \mathbf{\Sigma}^\top \mathbf{A})}{\lambda_{\max}(\mathbf{P})} > 0$. Then by the comparison lemma [78, Lemma 3.4],

$$V(\mathbf{y}(t)) \leq V(\mathbf{y}(t_0)) e^{-k_3(t-t_0)} \quad (3.26)$$

and as a consequence of (3.22),

$$\|\mathbf{y}(t)\| \leq \sqrt{\frac{k_2}{k_1}} \|\mathbf{y}(t_0)\| e^{-\frac{k_3}{2}(t-t_0)} \quad (3.27)$$

meaning that the origin is ES. If Assumption 2.3 holds globally, the bound in (3.16) also holds globally, and the origin is GES. $\square$

In the case with no external disturbance, the same analysis holds for the non-robust controller from [132] given by (2.40), (2.47). In other words, we can show global exponential stability results for the entire closed-loop system consisting of all the tasks, which could not be shown in [132].

Corollary 3.1. *Consider the closed-loop system consisting of N subsystems where the dynamics of subsystem i ($i = 1, \dots, N$) are given by (2.48). Let Assumptions 2.2 and 2.3 be satisfied. Then the origin of the system is ES. Furthermore, if Assumption 2.3 holds globally, the origin is GES.*

Proof. The proof follows the same structure as that of Theorem 3.1, but with no external disturbance term and no sign-term in the controller, i.e. $\mathbf{K}_{s,i} \equiv \mathbf{0}$, $\mathbf{f}_i \equiv \mathbf{0}$, $\forall i$. $\square$

Remark. The assumption that $\mathbf{J}^{\text{aug}}$ is invertible in the relevant part of the state-space is common in the literature [3, 44, 161], but restricts the choice of task combinations. In practice, choosing tasks which satisfy this assumption can be difficult and requires good planning, which is outside the scope of this thesis. However, the property of "global stability in the absence of singularities" indicates that convergence can be achieved for any magnitude of initial errors, so long as no singularities are encountered. This could not be concluded in [132] or for the tracking controller in [44], where theorems giving local stability properties were applied.

3.3 Comparison with existing control methods

The proposed robust controller (3.13) is very similar to the controllers proposed in [45, 132] and given in (2.40) with (2.47), and (2.53), respectively. All these controllers, as well as an additional previous robust controller proposed in [132] can be written on the same form, with the decoupling torque level control input (2.40) and a task level control force on the form

$$\mathbf{F}_{i,\text{ctrl}} = \mathbf{F}_{i,\text{ff}} + \mathbf{F}_{i,\text{comp}}, \quad (3.28)$$

where

$$\mathbf{F}_{i,\text{ff}} = \bar{\mathbf{M}}_i \ddot{\mathbf{x}}_{i,r} + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \dot{\mathbf{x}}_{i,r} + \gamma_i \begin{bmatrix} \dot{\mathbf{x}}_{i-1,d}^{\text{aug}} \\ \ddot{\mathbf{x}}_{i-1,d}^{\text{aug}} \end{bmatrix} \quad (3.29)$$

is a feedforward term and $\mathbf{F}_{i,\text{comp}}$ differs between the control approaches. For readability, they are summarized in Table 3.1.

As we can see, the different approaches appear quite similar, but with some distinct differences. The most obvious distinction is perhaps the inclusion of a sign-term in the robust controllers B, C, and D. These prioritize exact tracking, compared to the compliant behaviour of non-robust methods such as controllers A and E. In some cases, the passivity-motivated impedance behaviour exhibited by the continuous compliant controllers might be preferable, at the expense of lower tracking accuracy. However, using the robust method proposed in this chapter, disturbances with magnitudes below the values in $\mathbf{K}_{s,i}$ will be fully rejected, while larger forces will elicit impedance behaviour. This enables selective attenuation of disturbances,

while allowing impedance behaviour in response to larger forces through appropriate tuning of $\mathbf{K}_{s,i}$.

Compared with controller B from [45], given in (2.53), the controller proposed in this chapter includes the damping term $\mathbf{K}_{D,i}\mathbf{z}_i$ in place of $\mathbf{K}_{D,i}\mathbf{s}_i$, prioritizing convergence of the error $\mathbf{z}_i$ for the true task velocities over the error $\mathbf{s}_i$ for the dynamically consistent velocities. It also includes an extra proportional term $\mathbf{K}_{P,i}\tilde{\mathbf{x}}_i$ to improve the transient behaviour of lower-priority tasks. The controllers A, C, and D are analogous to backstepping control, due to the $\mathbf{z}_i$ -variables' similarity to a virtual backstepping control input. Compared with the previously proposed robust backstepping controller D, the sign-term of controller C proposed here is computed from the decoupled task velocities $\mathbf{s}_i$. Since the dynamics of the variables $\mathbf{z}_i$ depend on the acceleration of higher-priority tasks, applying the sign-term directly to $\mathbf{z}_i$ causes the discontinuous control input to act as a disturbance on lower-priority tasks, thus causing a build-up of chattering effects. The proposed approach avoids this issue of build-up by using the discontinuous term only in the dynamically decoupled velocities.

The controllers B and E were in [45, 54] proved to give asymptotic stability for the full stack of tasks, globally if there are no singularities in the workspace. Viewing the respective closed-loop systems as interconnected systems, it is possible to prove that these controllers too give exponential stability for the full stack of tasks. For completeness this proof is given in Section 3.A. Consequently, all the controllers A, B, C and E give equivalent stability properties of the respective closed-loop systems in the absence of disturbances.

3.4 Simulation results

This section presents simulations in which the proposed robust controller is compared with its continuous precursor, controller A, and the first-order sliding mode controller B (2.53) from [45] which has similar robustness properties. To isolate and emphasize the differences in convergence behaviour and transient responses among the control strategies without the added complexity introduced by a floating base, the simulations are performed on a model of a fixed-base manipulator.

Table 3.1: Compared control approaches

	Controller	$\mathbf{F}_{i,\text{comp}}$
A	Compliant [132] (2.47)	$-\mathbf{K}_{D,i}\mathbf{z}_i - \mathbf{K}_{P,i}\tilde{\mathbf{x}}_i$
B	First-order sliding mode [45] (2.53)	$-\mathbf{K}_{D,i}\mathbf{s}_i - \mathbf{K}_{s,i}\text{sgn}(\mathbf{s}_i)$
C	Proposed (3.13)	$-\mathbf{K}_{D,i}\mathbf{z}_i - \mathbf{K}_{P,i}\tilde{\mathbf{x}}_i - \mathbf{K}_{s,i}\text{sgn}(\mathbf{s}_i)$
D	Previous robust backstepping [132]	$-\mathbf{K}_{D,i}\mathbf{z}_i - \mathbf{K}_{P,i}\tilde{\mathbf{x}}_i - \mathbf{K}_{s,i}\text{sgn}(\mathbf{z}_i)$
E	Decoupled compliant [54] (2.51)	$-\mathbf{K}_{D,i}\mathbf{s}_i$

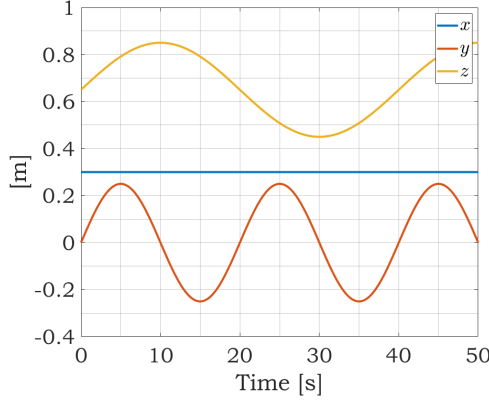


Figure 3.1: End-effector reference trajectory

Table 3.2: Controller gains

	Λ_i	$K_{D,i}$	K_i	$K_{s,i}$
$i = 1$	$5\mathbf{I}_3$	$15\mathbf{I}_3$	$10\mathbf{I}_3$	$5\mathbf{I}_3$
$i = 2$	$5\mathbf{I}_3$	$10\mathbf{I}_3$	$5\mathbf{I}_3$	$\frac{1}{2}\mathbf{I}_3$
$i = 3$	5	10	5	$\frac{1}{2}$

3.4.1 Simulation setup

The simulations are performed on a model of a Franka Emika Panda robot, a fixed-base manipulator with 7 joints, i.e. 7 DOFs. The simulation model is based on the dynamic model identified in [56], and implemented in MATLAB Simulink.

Three tasks are given. As task 1 the end-effector is to follow a trajectory tracing out a figure 8-like shape (lemniscate), shown in Figure 3.1. Task 2 is to hold the orientation of the end-effector fixed at the constant reference $\mathbf{x}_{2,d} = [\pi, 0, 0]^\top$ for roll, pitch, and yaw angles, corresponding to keeping the end-effector z -axis pointed downwards, and x -axis pointed straight forward. Finally, task 3 is to control the first, lowest joint to the constant reference $x_{3,d} = 0$. The task dimensions are $m_1 = 3$, $m_2 = 3$ and $m_3 = 1$. The initial configuration of the robot, its end-effector frame, desired end-effector orientation and the path to be traced out by the end-effector are shown in Figure 3.2. The initial joint configuration is $\boldsymbol{\theta}(0) = [\frac{\pi}{4}, -\frac{\pi}{3}, 0, -\frac{2\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{6}]^\top$. The control parameters are given in Table 3.2.

A constant disturbance force of 4 N in the positive y -direction acts on the end-effector from time $t = 30$ to $t = 40$ s, as indicated by the gray shading in the plots in the following subsection.

3.4.2 Results

The tracking errors are shown in Figures 3.3 to 3.5, and the velocity errors in Figures 3.3 to 3.5. The joint angles and torques are shown in Figures 3.9 and 3.10 for all three controllers. Note that the plots of the task errors are divided in two parts with different scale on the y -axis to display both the transient and the steady-state response. Figures 3.3 to 3.5 show the velocity tracking errors. For the lower priority

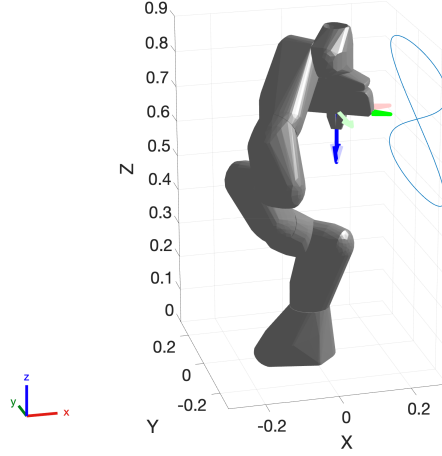


Figure 3.2: Illustration of initial configuration of the robot and the path to be traced by the desired end-effector trajectory. The end-effector frame is shown with x -axis in red, y -axis in green and z -axis in blue, respectively. The semi-transparent axes show the desired orientation.

tasks, i.e. for $i \geq 2$, the norms of $\mathbf{z}_i$ and $\mathbf{s}_i$ are plotted together, while for the highest priority task, the two velocity error variables $\mathbf{z}$ and $\mathbf{s}$ are equal, i.e. $\mathbf{z}_1 = \mathbf{s}_1$.

The magnitude of the disturbance is smaller than the elements of the gain matrix $\mathbf{K}_{s,1}$ in accordance with Assumption 3.1, meaning that the proposed robust controller with the chosen gains is expected to counteract the disturbance. Figure 3.3a shows that it does indeed successfully counteract the disturbance, and the tracking error in task 1 caused by the disturbance is smaller than 0.5 mm. The previous robust sliding mode controller performs equally well for the first task, as can be seen in Figure 3.5a. The compliant controller, however, is unable to fully counteract the disturbance, resulting in an offset of about 4.5 cm from the desired trajectory while the disturbance is present, as can be seen in Figure 3.4a. In addition, Figures 3.3 to 3.5 show that for all controllers the change in disturbance acting in the task space of task 1 disturbs the lower-priority tasks as well, as expected for hierarchical task-priority controllers.

Looking at the transients of the lower-priority tasks when using the proposed robust controller and the sliding mode controller, especially for task 2 it can be seen from Figures 3.3b, 3.3c, 3.5b and 3.5c that the proposed robust controller gives smaller overshoots and faster convergence than the previous robust controller.

Figures 3.9 and 3.10 show that both the joint angles and the torques are within the limits of what is feasible for the manipulator [50]. In physical experiments, it might still be desirable to saturate the control input to avoid the initial spike, at the cost of slower initial convergence. In Figure 3.9, it can be seen that the robust controllers give very high-frequency oscillations, called chattering, which is a common problem in high-gain control. Chattering in the control input is undesirable, as it causes more wear of the actuators, and it may not be possible for real-life actuators to realize it exactly. To alleviate chattering, one can for example use a continuous approximation of the sign-term, which was also considered in [45]. Another option

is to use higher-order sliding mode control methods, in which the signum term is integrated to give a continuous control input [139]. However, while the choice of damping term in $\mathbf{z}$ improved the performance of lower-priority tasks, it introduces top-down disturbances into the dynamics of $\mathbf{s}_i$ (3.8b), which may complicate the incorporation and analysis of higher-order sliding mode control.

Once the the velocity errors $\dot{\tilde{\mathbf{x}}}_j$ of higher-priority tasks are small, the difference between $\mathbf{s}_i$ and $\mathbf{z}_i$ is also small. This can be seen both in Figures 3.6b, 3.6c, 3.7b, 3.7c, 3.8b and 3.8c as well from the expression (3.4) relating $\mathbf{s}_i$ and $\mathbf{z}_i$. However, while the task variables are converging the proposed robust controller makes the error $\mathbf{z}_i$ converge faster than $\mathbf{s}_i$. Meanwhile the sliding mode controller gives the opposite effect, as can be seen by comparing Figures 3.6b and 3.6c with Figures 3.8b and 3.8c. This is due to the choice of which velocity error is used in the damping term; the proposed robust controller C (3.13) includes the term $-\mathbf{K}_{D,i}\mathbf{z}_i$, while the sliding mode controller B has the term $-\mathbf{K}_{D,i}\mathbf{s}_i$ instead, as given in Table 3.1. The improved transient behaviour of lower-priority tasks which could be observed when comparing Figures 3.3b and 3.3c with Figures 3.5b and 3.5c might be due to this prioritizing of the convergence of the error $\mathbf{z}_i$ for the true task velocities over the error $\mathbf{s}_i$ for the dynamically consistent velocities, which can be interpreted as a kind of relative velocities. However, comparing the expressions given in Table 3.1 for the first-order sliding mode controller B with the robust controller C proposed in this chapter, we see that the two controllers differ not only in the choice of damping term, but also in the addition of a proportional term $\mathbf{K}_{P,i}\tilde{\mathbf{x}}_i$ in the proposed controller (3.13). In order to further study the impact of the choice of damping term, an additional simulation was run with controller C (3.13) and with $\mathbf{K}_{P,i} = \mathbf{0}$ for all $i = 1, 2, 3$. The norm of the task errors for each of the three tasks is shown in Figure 3.11. It is plotted against the norm using the sliding mode controller, controller B, and the already discussed results with the proposed robust controller C and $\mathbf{K}_P$ as given in Table 3.2. For the first task, Figure 3.11a shows that the performance of all three controllers is very similar, but the controller proposed in this chapter gives slightly faster convergence. This is surprising as for the first tasks, since $\mathbf{z}_1 = \mathbf{s}_1$, the new proposed controller with $\mathbf{K}_P = \mathbf{0}$ and the first order sliding mode controller should give the same input for the first task. A possible explanation for the difference could be due to lower-priority tasks influencing the first task despite the decoupling input, which in turn could be due to the discontinuous input by the sign-terms. For the second task, shown in Figure 3.11b, the new proposed controller C performs better than the sliding mode controller B, regardless of $\mathbf{K}_P$. The addition of the proportional term improves the convergence very slightly, suggesting that it is the choice of $\mathbf{z}_i$ in the damping term that has the biggest impact for this lower-priority task. In contrast, for the third task shown in Figure 3.11c the fastest convergence is achieved by the first-order sliding mode controller. The third task is different in that the task variable is a joint angle, meaning that unlike the first two tasks, the execution of task 3 is performed by its own DOF directly rather than sharing DOFs with the other tasks. Under this circumstance, it may be that the first-order sliding mode controller has some beneficial properties. More than anything, the comparison shown in Figure 3.11 reveals that more investigation of the performance and individual properties of these controllers is needed.

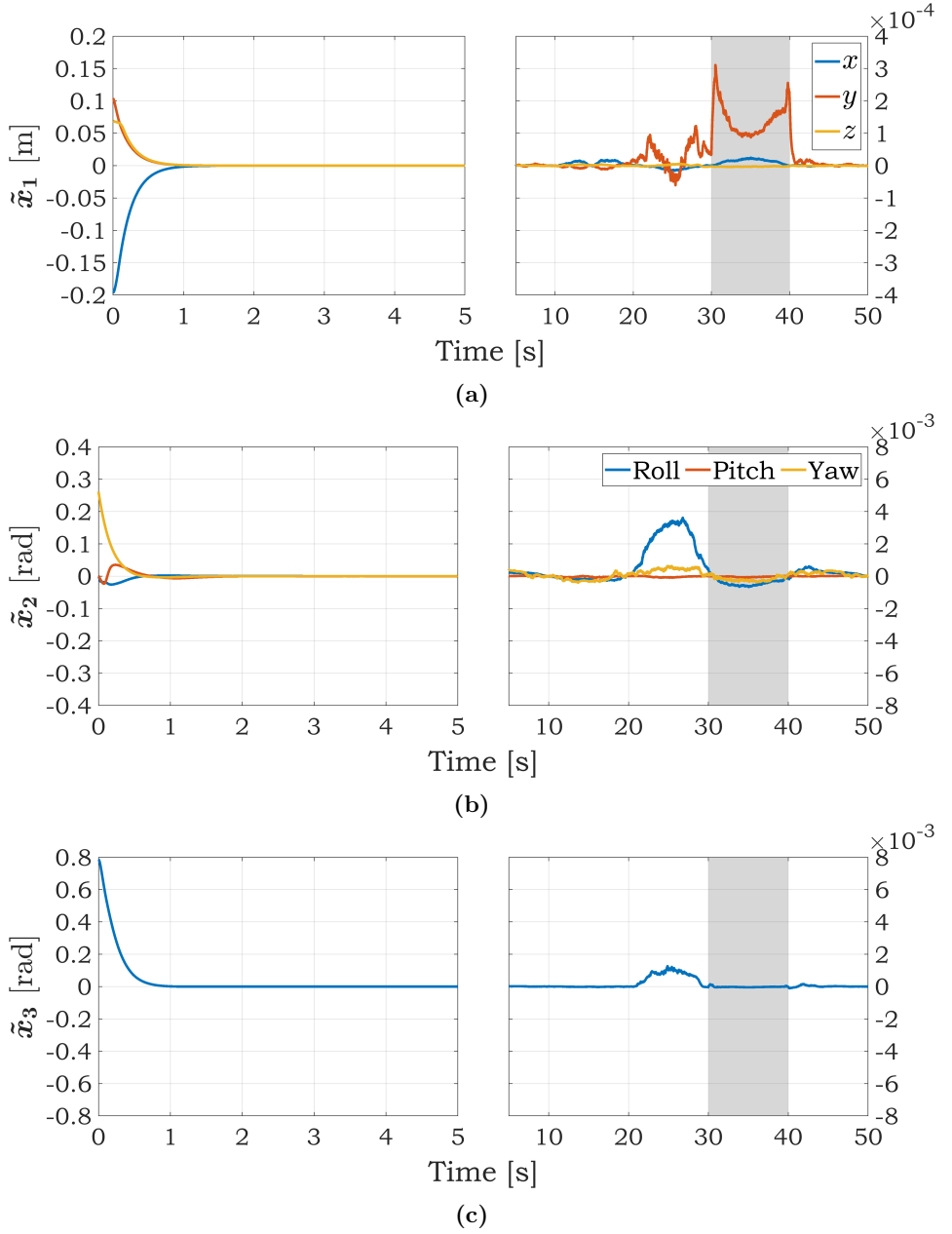


Figure 3.3: Task tracking errors of the proposed robust controller (C) in the presence of a disturbance

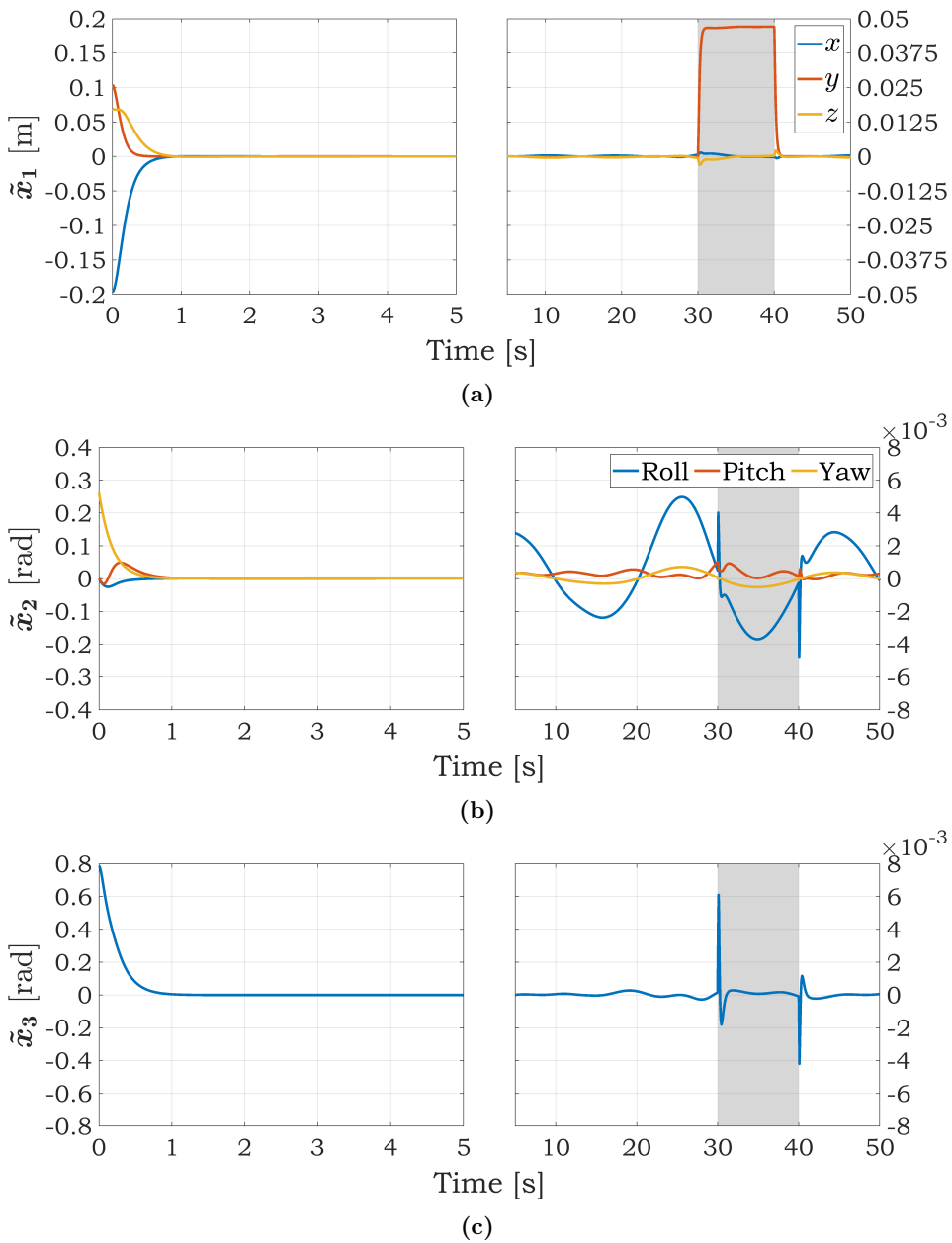


Figure 3.4: Task tracking errors of the continuous controller (A) in the presence of a disturbance

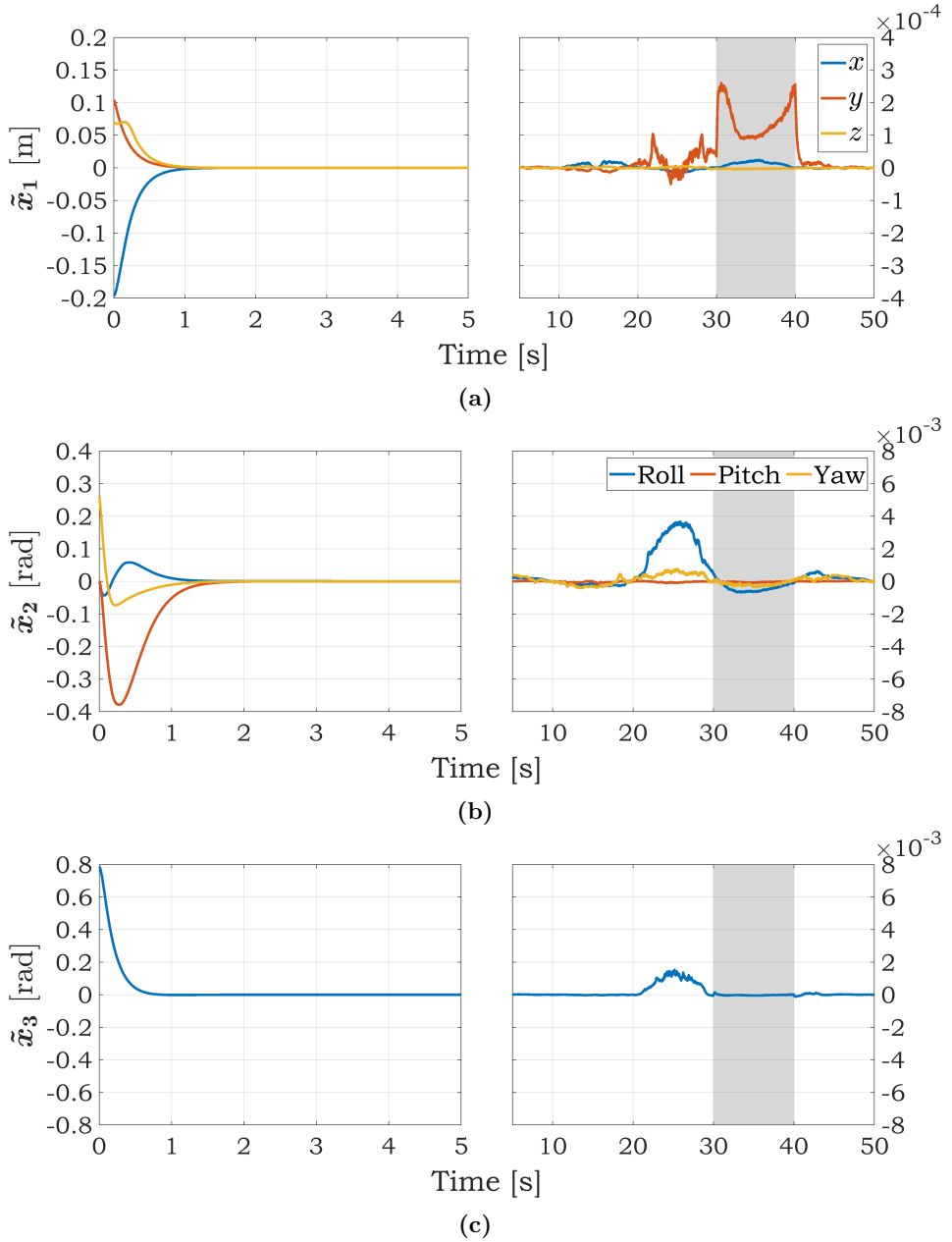


Figure 3.5: Task tracking errors of the first-order sliding mode controller (B) in the presence of a disturbance

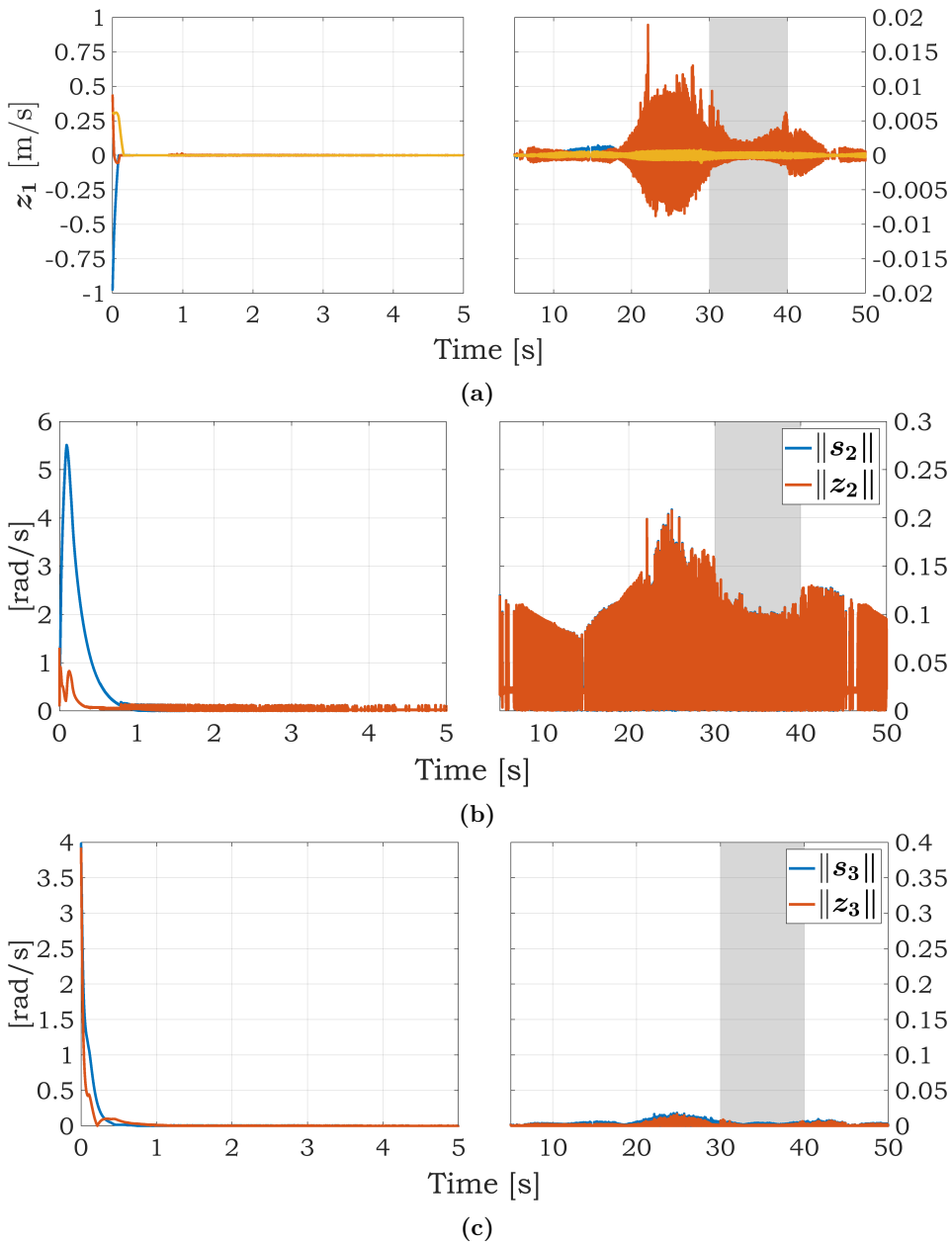


Figure 3.6: Task velocity errors of the proposed robust controller (C) in the presence of a disturbance

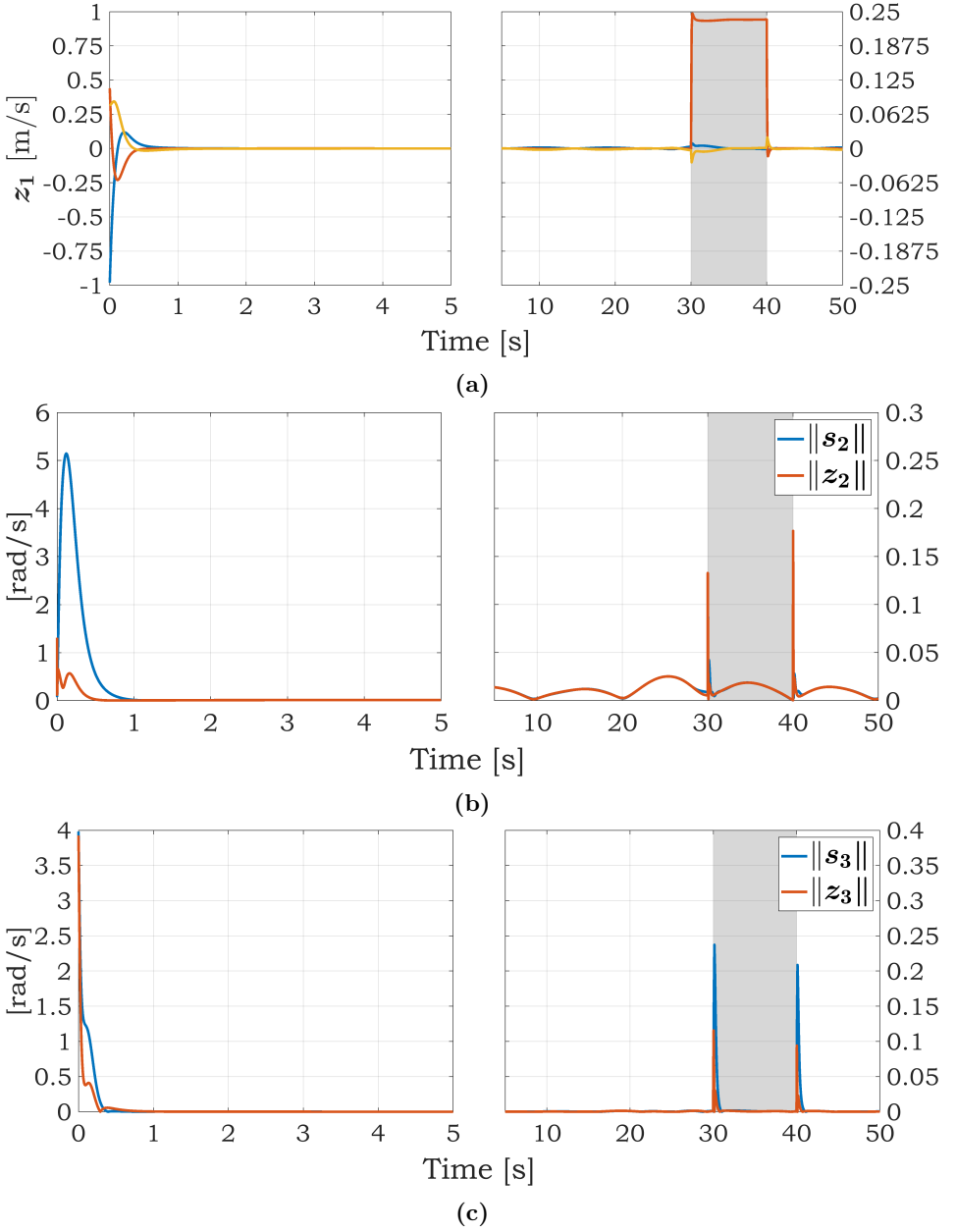


Figure 3.7: Task velocity errors of the continuous controller (A) in the presence of a disturbance

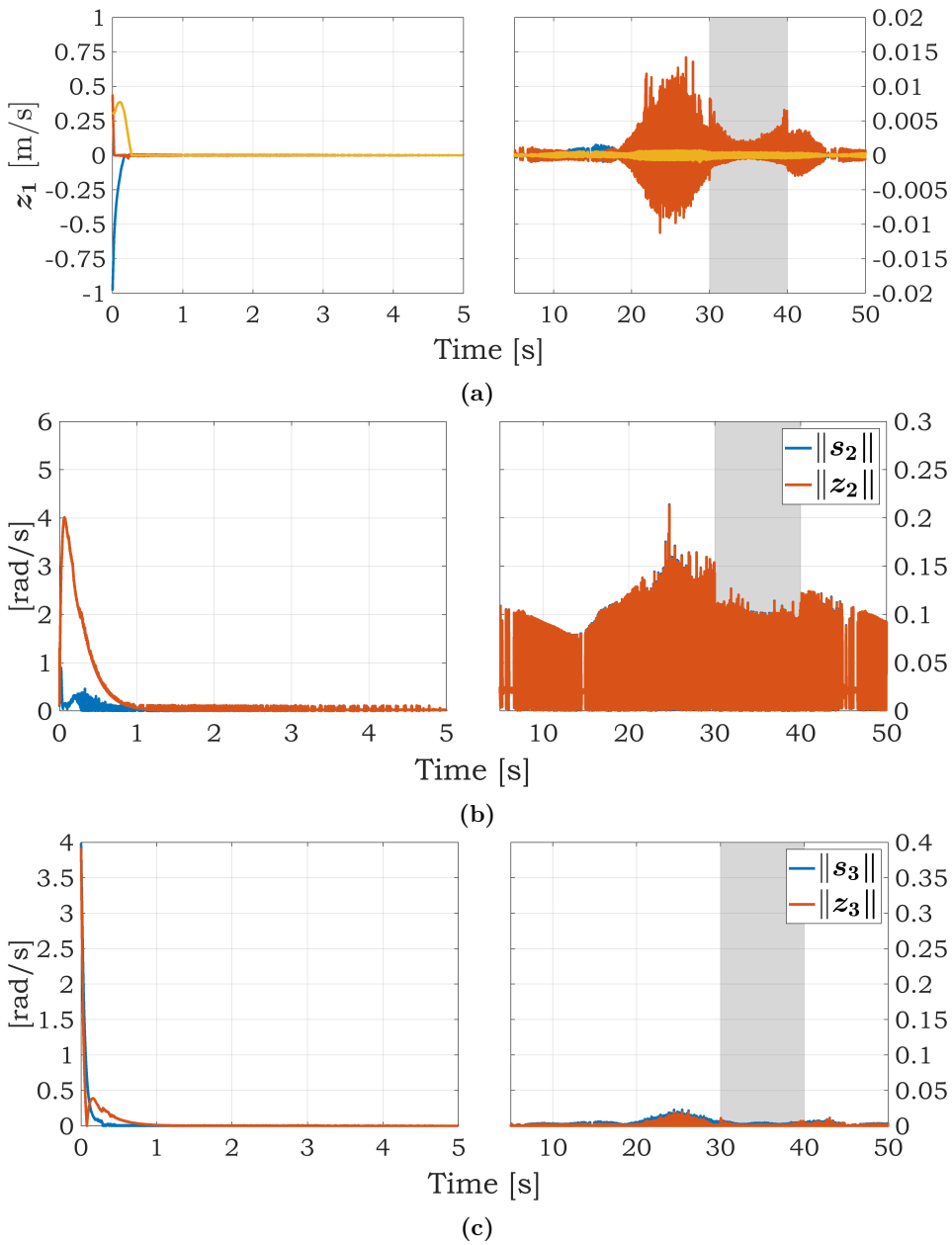


Figure 3.8: Task velocity errors of the first-order sliding mode controller (B) in the presence of a disturbance

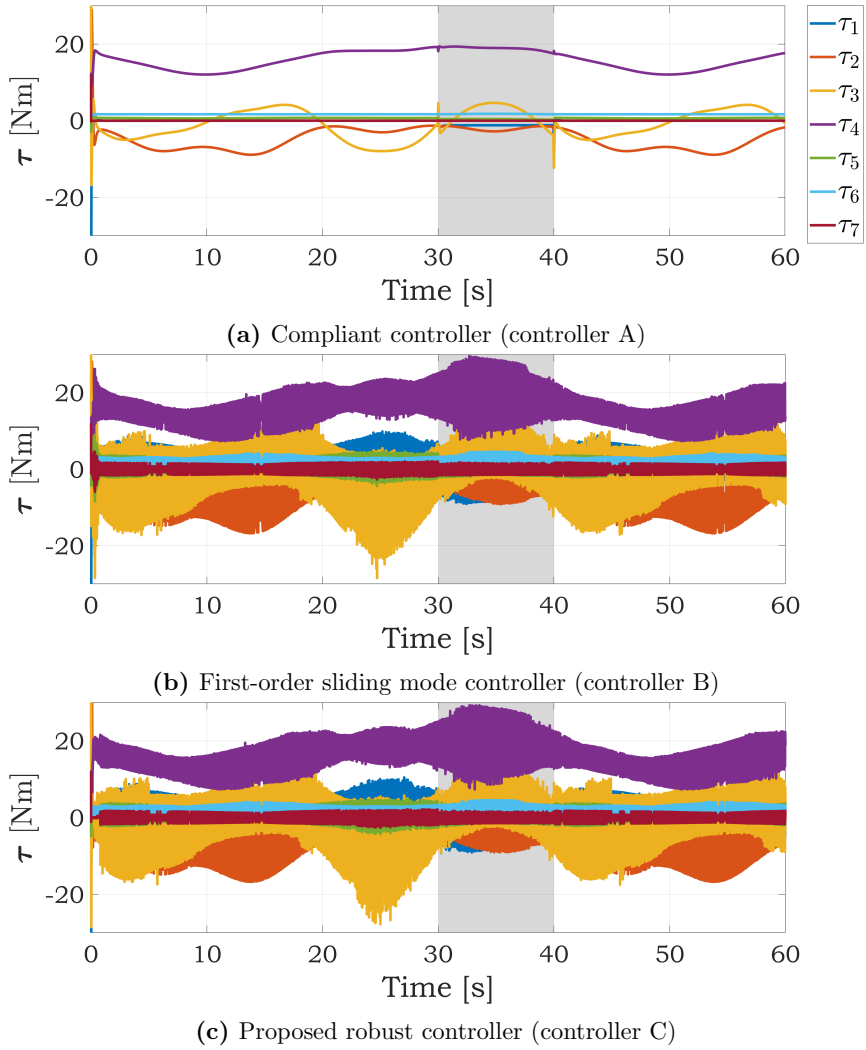


Figure 3.9: Joint torques applied by each of the compared controllers.

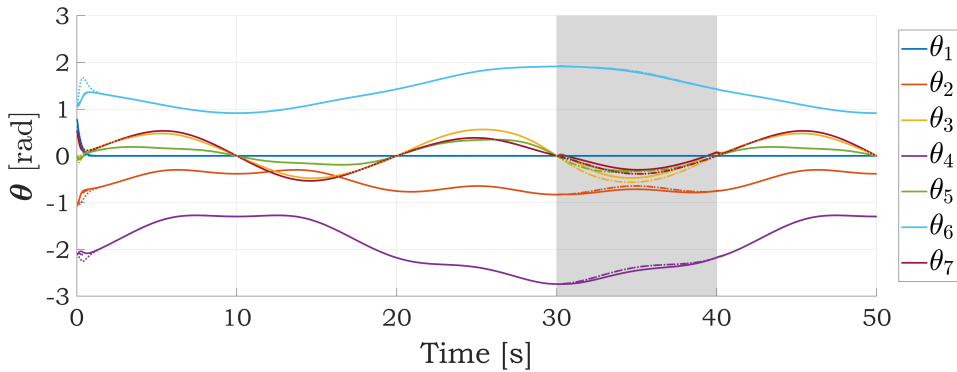


Figure 3.10: Joint angles for controller A (solid line), controller B (dotted line), and controller C (dashed line).

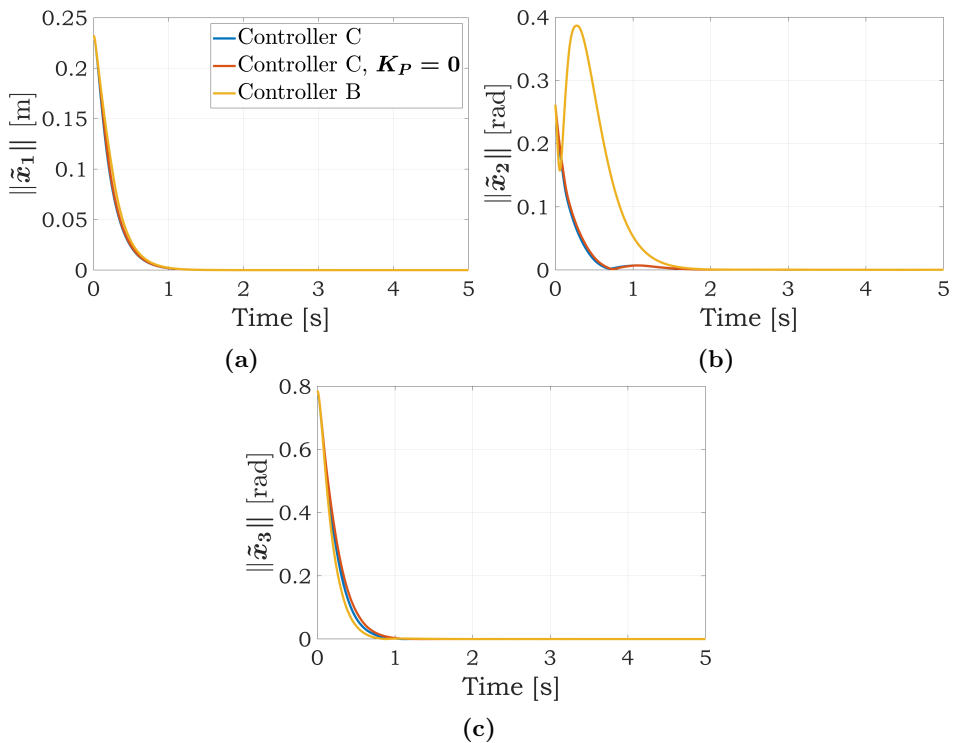


Figure 3.11: Norms of the task tracking errors, using the proposed controller (controller C), the proposed controller with $K_P = 0$, and the first order sliding mode controller (controller B).

3.5 Chapter summary

In this chapter a new, robust tracking control law was proposed for robot manipulators or VMSs performing multiple prioritized tasks, and shown to give exponential stability of all task tracking errors, and global exponential stability in the absence of any singularities. It was also shown how the same or similar stability analysis can be applied to three existing tracking control laws, for which stronger stability properties were established, improving upon previous results.

The performance of the proposed control law was investigated and compared against similar existing control laws through simulations. The simulation results show that the proposed control law exhibits comparable robustness to a previously proposed robust control law while offering improved convergence behaviour for lower-priority tasks. This suggests that using the original task velocities in the damping term can enhance the transient performance of lower-priority tasks.

3.A Exponential stability of existing controllers

For completeness and fairness of comparison, in this section we prove that the control law (2.51) from [54] also gives exponential stability of the origin.

The closed-loop dynamics (2.52) can be written into an interconnected system consisting of $2N$ subsystems, with subsystem number i for $i = 1, \dots, N$ given by

$$\bar{\mathbf{M}}_i \dot{\mathbf{s}}_i + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \mathbf{s}_i + \mathbf{K}_{D,i} \mathbf{s}_i = \mathbf{0}, \quad (3.30)$$

and subsystem number $N + i$ for $i = 1, \dots, N$ given by

$$\dot{\tilde{\mathbf{x}}}_i = -\boldsymbol{\Lambda}_i \tilde{\mathbf{x}}_i - \sum_{j=1}^{i-1} \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} \boldsymbol{\Lambda}_j \tilde{\mathbf{x}}_j + \mathbf{s}_i + \sum_{j=1}^{i-1} \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} \mathbf{s}_j. \quad (3.31)$$

For each subsystem, take the following Lyapunov function candidates:

$$\mathcal{V}_i(\mathbf{s}_i) = \frac{1}{2} \mathbf{s}_i^\top \bar{\mathbf{M}}_i \mathbf{s}_i \quad \text{for } i = 1, \dots, N, \quad (3.32a)$$

$$\mathcal{V}_{N+i}(\tilde{\mathbf{x}}_i) = \frac{1}{2} \|\tilde{\mathbf{x}}_i\|^2 \quad \text{for } i = 1, \dots, N. \quad (3.32b)$$

Differentiation of (3.32) gives

$$\begin{aligned} \dot{\mathcal{V}}_i &= \mathbf{s}_i^\top (-\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i}) \mathbf{s}_i - \mathbf{s}_i^\top \mathbf{K}_{D,i} \mathbf{s}_i + \frac{1}{2} \mathbf{s}_i^\top \dot{\bar{\mathbf{M}}}_i \mathbf{s}_i \\ &= -\mathbf{s}_i^\top (\boldsymbol{\delta}_{i,i} + \mathbf{K}_{D,i}) \mathbf{s}_i \leq -\lambda_{\min}(\boldsymbol{\delta}_{i,i} + \mathbf{K}_{D,i}) \|\mathbf{s}_i\|^2 \end{aligned} \quad (3.33)$$

and

$$\begin{aligned} \dot{\mathcal{V}}_{N+i} &= -\tilde{\mathbf{x}}_i^\top \boldsymbol{\Lambda}_i \tilde{\mathbf{x}}_i - \sum_{j=1}^{i-1} \tilde{\mathbf{x}}_i^\top \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} \boldsymbol{\Lambda}_j \tilde{\mathbf{x}}_j + \tilde{\mathbf{x}}_i^\top \mathbf{s}_i + \sum_{j=1}^{i-1} \tilde{\mathbf{x}}_i^\top \mathbf{J}_i \bar{\mathbf{J}}_j^{M+} \mathbf{s}_j \\ &\leq -\lambda_{\min}(\boldsymbol{\Lambda}_i) \|\tilde{\mathbf{x}}_i\|^2 + \sum_{j=1}^{i-1} \sigma_{\max}(\mathbf{J}_i \bar{\mathbf{J}}_j^{M+}) \lambda_{\max}(\boldsymbol{\Lambda}_j) \|\tilde{\mathbf{x}}_i\| \cdot \|\tilde{\mathbf{x}}_j\| \\ &\quad + \|\tilde{\mathbf{x}}_i\| \cdot \|\mathbf{s}_i\| + \sum_{j=1}^{i-1} \sigma_{\max}(\mathbf{J}_i \bar{\mathbf{J}}_j^{M+}) \|\tilde{\mathbf{x}}_i\| \cdot \|\mathbf{s}_j\|. \end{aligned} \quad (3.34)$$

Now let $\mathcal{V} = \sum_{i=1}^{2N} a_i^* \mathcal{V}_i$ where a_i^* , $i = 1, \dots, 2N$ are positive constants. Then

$$\dot{\mathcal{V}} \leq -\frac{1}{2} \boldsymbol{\phi}_2^\top (\mathbf{A}_2 \boldsymbol{\Sigma}_2 + \boldsymbol{\Sigma}_2^\top \mathbf{A}_2) \boldsymbol{\phi}_2 \quad (3.35)$$

where $\boldsymbol{\phi}_2 = [\|\mathbf{s}_1\|, \dots, \|\mathbf{s}_N\|, \|\tilde{\mathbf{x}}_1\|, \dots, \|\tilde{\mathbf{x}}_N\|]^\top$, and $\mathbf{A}_2 \in \mathbb{R}^{2N \times 2N}$ is a diagonal matrix with the coefficients a_i^* along its diagonal. The matrix $\boldsymbol{\Sigma}_2 \in \mathbb{R}^{2N \times 2N}$ can be

split into four $N \times N$ blocks, $\Sigma_2 = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$, with sub-blocks

$$\Sigma_{11} = \text{diag}(\lambda_{\min}(\delta_{1,1} + \mathbf{K}_{D,1}), \dots, \lambda_{\min}(\delta_{N,N} + \mathbf{K}_{D,N})), \quad (3.36a)$$

$$\Sigma_{12} = \mathbf{0}_{N \times N}, \quad (3.36b)$$

$$\Sigma_{21} = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ -\sigma_{21} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma_{N1} & -\sigma_{N2} & \cdots & -1 \end{bmatrix}, \quad (3.36c)$$

$$\Sigma_{22} = \begin{bmatrix} \lambda_{\min}(\mathbf{\Lambda}_1) & 0 & \cdots & 0 \\ -\sigma_{21}\lambda_{\max}(\mathbf{\Lambda}_1) & \lambda_{\min}(\mathbf{\Lambda}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma_{N1}\lambda_{\max}(\mathbf{\Lambda}_1) & -\sigma_{N2}\lambda_{\max}(\mathbf{\Lambda}_2) & \cdots & \lambda_{\min}(\mathbf{\Lambda}_N) \end{bmatrix}, \quad (3.36d)$$

where $\sigma_{ij} = \sigma_{\max}(\mathbf{J}_i \bar{\mathbf{J}}_j^{M+})$. The matrix Σ_2 is a lower triangular matrix with positive elements on the diagonal, and consequently all its leading principal minors are positive. Since all elements below the diagonal are non-positive, Σ_2 is an M-matrix, and therefore by Lemma 2.1 there exist constants a_i^* , $i = 1, \dots, 2N$ such that $(\mathbf{A}_2 \Sigma_2 + \Sigma_2^\top \mathbf{A}_2)$ is a positive definite matrix.

Finally, since we can write

$$\mathcal{V}(\mathbf{y}) = \frac{1}{2} \mathbf{y}^\top \begin{bmatrix} a_1 \mathbf{I}_{m_1} & 0 & \cdots & 0 & 0 \\ 0 & a_1 \bar{\mathbf{M}}_1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & a_N \mathbf{I}_{m_N} & 0 \\ 0 & 0 & \cdots & 0 & a_N \bar{\mathbf{M}}_N \end{bmatrix} \mathbf{y} \quad (3.37)$$

and since $\|\phi_2\| = \|\mathbf{y}\|$, similarly as in the proof of Theorem 3.1 it can be shown that $\mathcal{V}$ satisfies the conditions of [78, Theorem 4.10], and thus the origin of the system (2.52) is GES if Assumption 2.2 is satisfied and there are no singularities.

Remark. Exponential stability of the closed-loop system resulting from using the first-order sliding mode control law (2.53) from [45] can be proved along the same lines. However, exponential stability of this system could also be proved by viewing the resulting closed-loop system as a cascaded system and using the previously established fact that $\mathbf{s}$ is finite-time stable, which was proved by [45].

Chapter 4

Unifying the Generalized Jacobian Matrix with Prioritized Task Hierarchies

In this chapter, we demonstrate that the GJM coincides with the coordinate transformation into dynamically consistent coordinates for prioritized tasks introduced in Section 2.3, for a particular choice of tasks. This paves the way for extending methods such as those of [60] to an arbitrary number of tasks and tasks requiring tracking of time-varying trajectories, while preserving the beneficial division of actuator use in which thrusters are engaged only for repositioning the overall VMS. In order to achieve this also for VMSs equipped with thrusters on multiple links, such as AIAUVs, a thrust allocation algorithm is also developed in this chapter.

Contributions of this chapter

The contributions of this chapter can be summarized as follows. First, the GJM is shown to coincide with a dynamically consistent coordinate transformation for prioritized tasks for a particular choice of tasks, namely when choosing the highest priority task variable such that its velocity can be expressed as total momentum times multiplied by an invertible matrix. The coordinate transformation for such choices of highest priority tasks is shown to be upper block-triangular for an arbitrary number and choice of lower priority tasks, a structure which will be beneficial for control allocation. A tracking controller for multiple tasks, presented in [132], is applied to control the position of the CG, rotational momentum about CG and any choice of lower priority tasks for a UVMS, and shown to make the set in which all tracking errors are zero UAS. A control allocation algorithm using a $(1,2,3)$ -pseudoinverse is proposed for AIAUVs, whose thrusters can be placed on multiple links, to ensure that thrusters are used only to perform the highest priority task. Finally, the proposed control method consisting of tracking controller and control allocation for an AIAUV is validated in simulation.

The resulting control scheme proposed in this chapter generalizes the control scheme from [60]. Both [60] and its continuation [61] consider only control of end-effector pose in addition to CG position and momentum about it, and the VMSs are assumed to be non-redundant with respect to this choice of tasks. Furthermore,

only regulation to constant references is considered. The control scheme proposed in this chapter generalizes that of [60] to an arbitrary number and choice of lower priority tasks, as well as to tracking of time-varying trajectories, while still ensuring that thrusters are used only for positioning the CG and controlling the rotational momentum about it. The stability analysis performed in this chapter also demonstrates that the tracking controllers from [44, 132] can be used to perform tasks which do not fully determine the state of the VMS, so long as the set in which all task tracking errors are zero is compact.

Publications

This chapter is based on [159].

Organization of this chapter

This chapter is organized as follows: a brief introduction of related literature is given in Section 4.1. In Section 4.2, the relation between the GJM and the coordinate change to task coordinates from Section 2.3 is shown, and the structure of the mapping between task coordinates and the original system coordinates is examined. The GJM is combined with a tracking control law in Section 4.3, where the control allocation algorithm is also presented, and stability of the closed-loop system is analysed. Simulation results are shown in Section 4.4, and a summary of the chapter is given in Section 4.5.

4.1 Introduction

In this chapter, the GJM is unified and combined with the coordinate change and tracking controller introduced in Section 2.3, in order to reposition the workspace of light VMSs, such as AIAUVs, by moving their CG, and using thrusters for that only. This enables the AIAUV to function like a VMS where thrusters are used for gross motion of the overall vehicle and its workspace, while joints are used for more precise motion within the current workspace. The resulting control scheme is a generalization of the work from [60] to time-varying references, an arbitrary number and choice of secondary control objectives, and VMSs with actuators mounted on multiple links.

The literature on control of UVMSs has focused largely on VMSs with a large, heavy vehicle base, and control schemes in which coordination of the vehicle and manipulator is performed on a kinematic level only, while the dynamics of each are controlled separately [4]. Such an approach is satisfactory for these VMSs with a large, heavy base which does not experience much disturbance from the manipulator. However, free-floating VMSs with a comparably lighter vehicle base are more vulnerable to such disturbances, sometimes referred to as reaction forces [153], where the motion of the manipulator arm induces a motion of the vehicle base. This occurs because it is the CG of the vehicle that stays in place, while the joint motion changes the position of the vehicle base relative to the CG. For VMSs with a light base, the manipulator arm constitutes a larger part of their total mass, and therefore they are

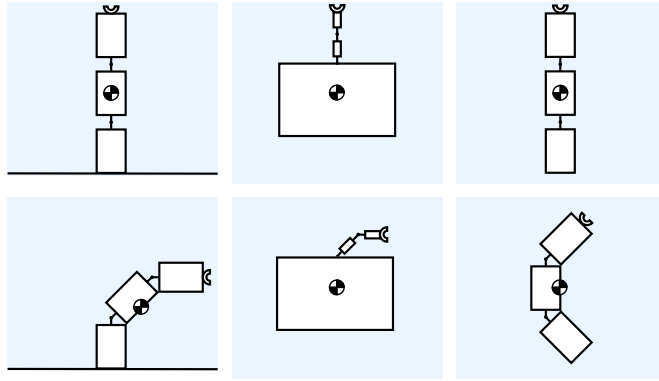


Figure 4.1: The same joint motion performed by (left to right): a fixed-base manipulator, a free-floating VMS with a heavy base, and a free-floating VMS with a light base.

more vulnerable to this reaction effect. The effect is illustrated in Figure 4.1, which shows the result of the same joint motion on a fixed-base manipulator, a free-floating VMS with a heavy base, and a free-floating VMS with a relatively light or no distinct base. Such effects were also observed by [20] in experiments with an AIAUV, in which joint motion at the front of the vehicle disturbed the pose of its backmost link, despite controlling its pose using a scheme designed to be robust against disturbances and uncertainties. This demonstrates the need to take these effects into account when designing control schemes for objectives requiring high precision.

The GJM was introduced by [153] to accurately control the end-effector of a free-floating VMS in space, despite the induced motion of the vehicle base due to joint motion. It is a means of including some knowledge of the system dynamics, specifically its inertia, into kinematic control schemes. The GJM is derived based on conservation of momentum, thanks to which the induced motion of the base can be calculated and compensated for. In [111], the GJM was used for impedance control at the end-effector, and both a velocity-level and torque-level control law is given. The latter is based on cancellation of dynamics in order to achieve the desired acceleration at the end-effector. Kinematic control using the GJM has also been applied to an AIAUV by [2] with promising simulation results.

The dynamics of floating-base robots were investigated by [52], where they are derived in terms of the total momentum and joint velocities as coordinates for any floating-base robots. This also includes legged robots, whose motion may be constrained by the necessity of maintaining contact with the ground. This choice of coordinates is shown to result in inertial decoupling of their dynamics, and the GJM is derived as a special case when total momentum is zero while the joints are used to perform an end-effector motion. The decoupling of dynamics in [52] is similar to the decoupling achieved through use of dynamically consistent null-space projections, like in Section 2.3, but the former considers only one set of specific coordinates for the system velocities. Coordinate transformations similar to the one from [52] were used by [58, 60, 61] to develop a series of dynamic control laws for regulation of the end-effector pose of VMSs in space, taking into account situations

in which total momentum of the system is not conserved. Thrusters are used to move the position of the CG of the system, but are not engaged to perform end-effector motion in order to conserve fuel. Manipulators which are redundant with respect to the task of controlling the end-effector pose were considered in [58]. The null-space velocities, i.e. manipulator motion which does not contribute to neither end-effector velocity nor total momentum, are dynamically decoupled from the end-effector motion so that they do not disturb it, but are not employed to perform any additional tasks. A similar approach has also been applied to an aerial VMS consisting of a helicopter with a comparably heavy arm attached to it in [53], to ensure that control objectives to be performed by the manipulator arm do not disturb the position of the CG of the overall system and the attitude of the base vehicle.

In the previous work applying the GJM to control an AIAUV [2], only joint motion is considered within what effectively becomes a stationary workspace, since thrusters are not used. On the other hand, in previous works such as [45, 132], where both thrusters and joints are utilized to have an AIAUV perform multiple tasks, the allocation of control inputs between thrusters or joint torques has not received much attention. AUVs, including AIAUVs, are typically actuated by propeller-based thrusters, which are subject to complex dynamics, especially during maneuvers with unsteady flow over the propellers [65]. In addition, these dynamics are often not included in the models used for control development, resulting in unmodeled input delays and uncertainties. Therefore, it would be beneficial to use thrusters only for gross motion to reposition the workspace of a UVMS, while the joints are used for precise motions during inspection and intervention tasks. In addition, choosing the CG as the point to reposition reduces the need for using thrusters to counteract disturbances from the joint motion, which may contribute to reducing energy consumption and extend mission duration.

4.2 The GJM and task hierarchies

In this section, the GJM is derived as a special case of the coordinate transformation covered in Section 2.3. Specifically, it is shown that the GJM is the projected Jacobian of an end-effector pose task when the primary task belongs to a family of momentum-like tasks. Finally, we show that the beneficial structure of the coordinate transformation matrix obtained by [60] can be preserved for any choice of lower-priority tasks.

Let $\mathbf{h} \in \mathbb{R}^6$ be the total momentum about a frame c with origin at the CG of the VMS and a fixed orientation relative to the inertial frame, and given in [60] as

$$\mathbf{h} = \mathbf{Ad}_{cb}^{-\top} [\mathbf{M}_b \quad \mathbf{M}_{bm}] \boldsymbol{\zeta} \quad (4.1)$$

where $\mathbf{Ad}_{cb}$ denotes the adjoint of the transformation describing the pose of the base frame given in the c -frame. It is given by

$$\mathbf{Ad}_{cb} = \begin{bmatrix} \mathbf{R}_{cb} & \mathbf{S}(\mathbf{p}_{cb})\mathbf{R}_{cb} \\ \mathbf{0}_{3 \times 3} & \mathbf{R}_{cb} \end{bmatrix} \quad (4.2)$$

where $\mathbf{R}_{cb}$ describes the orientation of the base frame relative to the c -frame, and $\mathbf{p}_{cb}$ describes the position of the base frame relative to the c -frame. Now let the primary

task to be performed have a task velocity which can be written as

$$\mathbf{v}_1 = \mathbf{X}(\boldsymbol{\eta}, \boldsymbol{\theta}) \mathbf{h} \quad (4.3)$$

where $\mathbf{X}(\boldsymbol{\eta}, \boldsymbol{\theta}) \in \mathbb{R}^{6 \times 6}$ is invertible in the considered domain. We will refer to tasks with velocity as given by (4.3) as momentum-like.

Remark. Such tasks include the velocity of the CG together with rotational momentum as in [60], or a generalized average velocity, the latter of which was used by [61] to decompose the total end-effector velocity.

By inserting (4.1) into (4.3) we obtain

$$\mathbf{v}_1 = \mathbf{X} \mathbf{A} \mathbf{d}_{cb}^{-\top} [\mathbf{M}_b \quad \mathbf{M}_{bm}] \boldsymbol{\zeta} = \mathbf{J}_1 \boldsymbol{\zeta}. \quad (4.4)$$

We can then find

$$\begin{aligned} \mathbf{N}_2 &= \mathbf{I}_{6+n} - \mathbf{J}_1^\top \mathbf{J}_1^{M+, \top} \\ &= \mathbf{I}_{6+n} - \begin{bmatrix} \mathbf{M}_b \\ \mathbf{M}_{bm}^\top \end{bmatrix} \mathbf{A} \mathbf{d}_{cb}^{-1} \mathbf{X}^\top [\mathbf{X}^{-\top} \mathbf{A} \mathbf{d}_{cb} \mathbf{M}_b^{-1} \quad \mathbf{0}_{6 \times n}] \\ &= \begin{bmatrix} \mathbf{0}_{6 \times 6} & \mathbf{0}_{6 \times n} \\ -\mathbf{M}_{bm}^\top \mathbf{M}_b^{-1} & \mathbf{I}_n \end{bmatrix}. \end{aligned} \quad (4.5)$$

It can be verified that $\mathbf{N}_2^\top$ is equal to a matrix selecting only the contribution from the joint motion based on the coordinate transformation of [52]. Consequently, $\mathbf{N}_2^\top$ can be interpreted as a projection into the space of what is called the internal motion of the system by [60].

4.2.1 The GJM as a prioritized Jacobian

Let there be a second task $\mathbf{x}_2 \in \mathbb{R}^{m_2}$ whose Jacobian can be split into two parts corresponding to the contributions from base velocity and joint velocities as in [153]:

$$\dot{\mathbf{x}}_2 = \mathbf{J}_2 \boldsymbol{\zeta} = [\mathbf{J}_{2,b} \quad \mathbf{J}_{2,m}] \boldsymbol{\zeta} \quad (4.6)$$

with $\mathbf{J}_{2,b} \in \mathbb{R}^{m_2 \times 6}$, $\mathbf{J}_{2,m} \in \mathbb{R}^{m_2 \times n}$. We then have

$$\bar{\mathbf{J}}_2 = \mathbf{J}_2 \mathbf{N}_2^\top = [\mathbf{0}_{6 \times 6} \quad \mathbf{J}_{2,m} - \mathbf{J}_{2,b} \mathbf{M}_b^{-1} \mathbf{M}_{bm}] \quad (4.7)$$

where $\mathbf{J}_{2,m} - \mathbf{J}_{2,b} \mathbf{M}_b^{-1} \mathbf{M}_{bm}^\top$ is the GJM [153]. As a consequence, if the second task is chosen to be the end-effector pose, the dynamically consistent task velocity $\mathbf{v}_2$ can be interpreted as the so-called internal end-effector velocity [60].

4.2.2 Triangular structure of coordinate transformation

Since $\mathbf{N}_{i-1}$ is a projection matrix, it has to satisfy $\mathbf{N}_{i-1} = \mathbf{N}_{i-1}^2$, and we can write (2.31) for $i = 2, \dots, N$ as

$$\mathbf{N}_i = \mathbf{N}_{i-1} \left(\mathbf{I}_{6+n} - \bar{\mathbf{J}}_{i-1}^\top \bar{\mathbf{J}}_{i-1}^{M+, \top} \right). \quad (4.8)$$

From the expression (4.8), it is apparent that every prioritized Jacobian (2.30) of lower priority tasks, i.e. with $i \geq 2$, will be multiplied by $\mathbf{N}_2^\top$, since for $i \geq 3$ we can write

$$\bar{\mathbf{J}}_i = \mathbf{J}_i \left(\prod_{j=2}^{i-1} \mathbf{I}_{6+n} - \bar{\mathbf{J}}_j^\top \bar{\mathbf{J}}_j^{M+, \top} \right)^\top \mathbf{N}_2^\top. \quad (4.9)$$

If the primary task belongs to the group of "momentum-like" tasks whose velocity can be described by (4.3), the expression for $\mathbf{N}_2$ given by (4.5) ensures that the Jacobians $\bar{\mathbf{J}}_i$ with $i \geq 2$ all start with six columns of zeros. As a result, $\bar{\mathbf{J}}$ has the following block-triangular structure:

$$\bar{\mathbf{J}} = \begin{bmatrix} \mathbf{X} \mathbf{A} d_{cb}^{-\top} \mathbf{M}_b & \mathbf{X} \mathbf{A} d_{cb}^{-\top} \mathbf{M}_{bm} \\ \mathbf{0}_{n \times 6} & \bar{\mathbf{J}}_\theta \end{bmatrix} \quad (4.10)$$

where $\bar{\mathbf{J}}_\theta \in \mathbb{R}^{n \times n}$ consists of the last n columns of the Jacobians $\bar{\mathbf{J}}_i$, $i = 2, \dots, N$, stacked. If $\bar{\mathbf{J}}$ is invertible, so is $\bar{\mathbf{J}}_\theta$. Thus the triangular structure of the coordinate transformation matrix from [60] can be achieved for any number and choice of lower-priority tasks when combined with a momentum-like primary task. As a result of this structure, only the joint velocities contribute to the dynamically consistent task velocities of the lower-priority tasks. If the inverse mapping, i.e. the mapping from desired inputs in dynamically consistent task coordinates to the actuators, is correspondingly lower block-triangular, then only the primary task engages the thrusters, while inputs to the lower-priority tasks are realized using only the joint torques. If the input configuration matrix (2.5) is block-diagonal or lower block-triangular and square, this follows directly when $\bar{\mathbf{J}}$ is upper block-triangular. Otherwise, the control allocation must be done in a way which preserves the desired triangular structure of the inverse mapping. This is presented in Subsection 4.3.2.

4.3 Task tracking control and control allocation for UVMSs

In this section we use the results of Section 4.2 to design a tracking control law and control allocation algorithm which use thrusters only for overall gross motion of the system to reposition its workspace, while the joints are used to perform additional tasks within the workspace. The tracking control law is based on the control law from [132], given in Section 2.3.2, with some required modifications, and we analyse the stability of the resulting closed-loop system. Moreover, a control allocation algorithm which ensures a triangular structure of the mapping from virtual task inputs to actuators is presented.

The control law will be applied to a UVMS with dynamic model as described in Section 2.1.1. Since the considered vehicle has only revolute joints, the joint angles $\boldsymbol{\theta}$ can be viewed as an element of the n -torus $\mathbb{T}^n = \mathbb{S}^1 \times \mathbb{S}^1 \times \dots \times \mathbb{S}^1$. In this chapter, we consider vehicles with actuator configuration matrix $\mathbf{B}_{\text{thr}}(\boldsymbol{\theta})$ (2.5) satisfying the following assumption:

Assumption 4.1. The vehicle is fully actuated at all times, i.e. $\text{rank}(\mathbf{B}_{\text{thr}}(\boldsymbol{\theta})) = 6 \forall \boldsymbol{\theta}$.

Remark. Assumption 4.1 is an assumption that the UVMS is well-designed, i.e. equipped with a sufficient number of actuators to ensure controllability of the base motion in any configuration of the UVMS.

Remark. For a UVMS with all thrusters placed on the base, $\mathbf{B}_\theta(\boldsymbol{\theta}) = 0 \forall \boldsymbol{\theta}$.

As our primary task, like [60] we choose to control the position of the CG, denoted $\mathbf{p}_{\text{cg}}$, and the total rotational momentum about a frame located at the CG and with axes which are fixed relative to the inertial axes. In order to relate the velocity of the CG to the total momentum in an underwater setting, we make the following assumption in the control design and analysis:

Assumption 4.2. The added mass effects are negligible.

Remark. The presence of added mass effects will result in a disturbance which is not compensated directly by the controller, nor taken into account in the analysis. Since knowledge of the added mass effects will be approximate at best, and unnecessarily complicate the notion of controlling the center of gravity, we let this disturbance be dealt with by the general robustness of the controller. This will be further discussed in connection with simulation results in Section 4.4.

Remark. An alternative to Assumption 4.2 allowing for including an approximation of the added mass effects is given in Section 4.3.3.

The task velocity $\mathbf{v}_1 = [\dot{\mathbf{p}}_{\text{cg}}^\top, \dot{\mathbf{h}}_{\text{rot}}^\top]^\top$, where $\mathbf{h}_{\text{rot}}$ is the rotational momentum, is given by (4.3) with

$$\mathbf{X} = \begin{bmatrix} \frac{1}{m_{\text{tot}}} \mathbf{I}_3 & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_3 \end{bmatrix} \quad (4.11)$$

where m_{tot} is the total mass of the VMS. The references for the position of the CG and lower-priority tasks are in general considered to be time-varying, while for the momentum the goal is to drive it to zero. Thus we have no task variable $\mathbf{x}_1$ such that $\dot{\mathbf{x}}_1 = \mathbf{v}_1$. Instead we describe the primary task using only $\mathbf{p}_{\text{cg}}$ and $\mathbf{v}_1$. We then have $N - 1$ additional tasks of the form described in Section 2.3.

4.3.1 Tracking control law

The tracking control law from [132], introduced in Section 2.3.2, will be modified to accommodate the choice of primary task.

Applying the coordinate transformation from Section 2.3 to the UVMS dynamics (2.1b) and taking into account the actuator mapping, the transformed dynamics (2.37) become

$$\bar{\mathbf{M}}\dot{\mathbf{v}} + \boldsymbol{\mu}\mathbf{v} + \boldsymbol{\delta}\mathbf{v} = \bar{\mathbf{J}}^{-\top} (\mathbf{B}(\boldsymbol{\theta})\mathbf{u} - \mathbf{g}). \quad (4.12)$$

Let

$$\bar{\mathbf{J}}^{-\top} \mathbf{B}(\boldsymbol{\theta})\mathbf{u} = \mathbf{F} \quad (4.13)$$

where we treat $\mathbf{F}$ as a virtual input to the dynamically consistent task velocities. The control allocation problem of finding the corresponding control inputs $\mathbf{u}$ will be solved later in Section 4.3.2. Similarly to the decoupling control input (2.40), we select

$$\mathbf{F} = \bar{\mathbf{J}}^{-\top} \mathbf{g} + \begin{bmatrix} \mathbf{F}_{\mu,1} \\ \vdots \\ \mathbf{F}_{\mu,N} \end{bmatrix} + \begin{bmatrix} \mathbf{F}_{\delta,1} \\ \vdots \\ \mathbf{F}_{\delta,N} \end{bmatrix} + \begin{bmatrix} \mathbf{F}_{1,\text{ctrl}} \\ \vdots \\ \mathbf{F}_{N,\text{ctrl}} \end{bmatrix} \quad (4.14)$$

where $\mathbf{F}_{i,\text{ctrl}}$ is a control input for task i like in Section 2.3.2. The terms $\mathbf{F}_{\mu,i}$ and $\mathbf{F}_{\delta,i}$, $i = 1, \dots, N$, are the terms cancelling out parts of the dynamics in order to fully decouple the dynamics of each task. They are chosen to be

$$\mathbf{F}_{\delta,i} = \sum_{j=1}^{i-1} \delta_{i,j} \mathbf{v}_j + \sum_{j=i+1}^N \delta_{i,j} \mathbf{v}_j \quad (4.15)$$

which corresponds to

$$\begin{bmatrix} \mathbf{F}_{\delta,1} \\ \vdots \\ \mathbf{F}_{\delta,N} \end{bmatrix} = \bar{\mathbf{J}}^{-\top} \boldsymbol{\tau}_{\delta} \quad (4.16)$$

with $\boldsymbol{\tau}_{\delta}$ from (2.41b), and

$$\mathbf{F}_{\mu,i} = \begin{cases} \mathbf{0}_{6 \times 1} & \text{for } i = 1 \\ \sum_{j=1}^{i-1} \boldsymbol{\mu}_{i,j} \mathbf{v}_j + \sum_{j=i+1}^N \boldsymbol{\mu}_{i,j} \mathbf{v}_j & \text{for } i = 2, \dots, N \end{cases} \quad (4.17)$$

The control law (4.14) is almost equivalent to setting $\mathbf{F} = \bar{\mathbf{J}}^{-\top} \boldsymbol{\tau}$ with $\boldsymbol{\tau}$ as given by (2.40), except for the term (4.17). For $\mathbf{X}$ constant, and following along the same reasoning as [52], namely that momentum is conserved in the absence of external forces, it can be shown that the first 6 rows of the term $\boldsymbol{\mu} \mathbf{v}$ are equal to 0 in (4.12). Thus no cancellation of these terms for the first task is necessary. Finding $\mathbf{F}_{\mu,1}$ using the same expression as $\mathbf{F}_{\mu,i}$ for $i = 2, \dots, N$ recovers the control law (2.40) from [132].

As in Section 2.3.2, inserting (4.14) into (4.12) fully decouples the dynamics of the dynamically consistent task-velocities. For the lower priority tasks, i.e. for with $i \geq 2$, the virtual control inputs $\mathbf{F}_{i,\text{ctrl}}$ are chosen as (2.47).

Control input for the CG and momentum task For the primary task, we construct a modified reference velocity $\dot{\mathbf{p}}_{\text{cg},r}$ for the position of CG analogously to (2.45):

$$\dot{\mathbf{p}}_{\text{cg},r} = \dot{\mathbf{p}}_{\text{cg},d} - \boldsymbol{\Lambda}_{1,p} \tilde{\mathbf{p}}_{\text{cg}} \quad (4.18)$$

and set

$$\mathbf{z}_1 = \begin{bmatrix} \dot{\mathbf{p}}_{\text{cg}} - \dot{\mathbf{p}}_{\text{cg},r} \\ \mathbf{h}_{\text{rot}} \end{bmatrix} \quad (4.19)$$

The control input for the primary task is then chosen as

$$\mathbf{F}_{1,\text{ctrl}} = \bar{\mathbf{M}}_1 \begin{bmatrix} \ddot{\mathbf{p}}_{\text{cg},r} \\ \mathbf{0} \end{bmatrix} + (\boldsymbol{\mu}_{1,1} + \boldsymbol{\delta}_{1,1}) \begin{bmatrix} \dot{\mathbf{p}}_{\text{cg},r} \\ \mathbf{0} \end{bmatrix} - \mathbf{K}_{D,1} \mathbf{z}_1 - \begin{bmatrix} \mathbf{K}_{P,1,p} \tilde{\mathbf{p}}_{\text{cg}} \\ \mathbf{0} \end{bmatrix} \quad (4.20)$$

where $\mathbf{K}_{D,1} = \begin{bmatrix} \mathbf{K}_{D,1,p} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{D,1,h} \end{bmatrix}$.

The choice (4.20) corresponds to (2.47) with $\mathbf{h}_{\text{rot},d} = \mathbf{0}$, and inserting $[\tilde{\mathbf{p}}_{\text{cg}}^{\top}, \mathbf{0}^{\top}]^{\top}$ in place of $\tilde{\mathbf{x}}_1$. When added mass effects are neglected in the mass matrix $\mathbf{M}$ as stated in Assumption 4.2, the task inertia $\bar{\mathbf{M}}_1$ for this primary task becomes block-diagonal as in [60], and the upper left 3×3 block of $\boldsymbol{\mu}_{1,1}$ consists of zeros.

Remark. If the reference for the position of CG is also chosen constant, and in the absence of external damping or hydrostatic forces, the control law given by (4.14), (4.20) for the momentum task reduces to the same one as in [60] with total proportional gain $\mathbf{K}_{D,1,p}\mathbf{A}_{1,p} + \mathbf{K}_{P,1,p}$. If the only lower-priority task is chosen to be regulation of end-effector pose as in the scenario considered by [60], the complete input (4.14), (2.47) for the end-effector task differs only by the cancellation of Coriolis terms. This is not done by [60], where since only two tasks with constant references are considered, the disturbance from the first task through the Coriolis term will vanish when the first task is completed.

4.3.2 Control allocation

To ensure that thrusters are used for gross motion of the workspace, while joints are used for more precise motion within the current workspace, the control allocation must be designed to preserve the desired triangular structure when inverting the mapping (4.13) in order to map the virtual inputs $\mathbf{F}$ to actuator inputs $\mathbf{u}$. As stated in Section 4.2, if the input configuration matrix $\mathbf{B}(\boldsymbol{\theta})$ is block-diagonal, this follows from the block-triangular structure of $\bar{\mathbf{J}}$. However, for an AIAUV, the matrix $\mathbf{B}(\boldsymbol{\theta})$ given by (2.5) is lower block-triangular, and when $\bar{\mathbf{J}}$ has the form (4.10), the product $\bar{\mathbf{J}}^{-\top}(\boldsymbol{\xi})\mathbf{B}(\boldsymbol{\theta})$ is likewise lower block triangular. As a consequence, joint torques have no impact on the primary task as intended. In order to also ensure that thrusters are engaged only to realize the input to the primary task, the inverse of the mapping $\bar{\mathbf{J}}^{-\top}(\boldsymbol{\xi})\mathbf{B}(\boldsymbol{\theta})$ must also be lower block-triangular. However, the pseudoinverse of a block-triangular matrix is not necessarily block-triangular [103].

Based on [103], for a block-triangular matrix $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, we define

$$\mathbf{A}^\ddagger \triangleq \begin{bmatrix} \mathbf{A}_{11}^\dagger & \mathbf{0} \\ -\mathbf{A}_{22}^\dagger \mathbf{A}_{21} \mathbf{A}_{11}^\dagger & \mathbf{A}_{22}^\dagger \end{bmatrix} \quad (4.21)$$

The matrix $\mathbf{A}^\ddagger$ is a so-called (1,2,3)-pseudoinverse of $\mathbf{A}$ if (and only if) the range of the matrix $\mathbf{A}_{21}$ is a subset of the range of $\mathbf{A}_{22}$ [103]. If $\mathbf{A}_{22}$ is square and has full rank, this condition is satisfied trivially. A (1,2,3)-pseudoinverse possesses three out of four of the properties of the usual pseudoinverse, the exception being that $(\mathbf{A}^\ddagger \mathbf{A})^\top \neq \mathbf{A}^\ddagger \mathbf{A}$. If, in addition, the range of $\mathbf{A}_{21}^\top$ is a subset of the range of $\mathbf{A}_{11}^\top$, then $\mathbf{A}^\ddagger = \mathbf{A}^\dagger$.

We solve (4.13) for $\mathbf{u}$ by allocating the control inputs according to

$$\mathbf{u} = (\bar{\mathbf{J}}^{-\top} \mathbf{B}(\boldsymbol{\theta}))^\ddagger \mathbf{F}. \quad (4.22)$$

By Assumption 4.1, $\mathbf{B}_{\text{thr}}$ has full row rank so that we have $\mathbf{B}_{\text{thr}} \mathbf{B}_{\text{thr}}^\dagger = \mathbf{I}_6$. When, as a consequence of Assumption 2.3, $\bar{\mathbf{J}}$ also has full rank, it can be shown that

$$(\bar{\mathbf{J}}^{-\top} \mathbf{B}(\boldsymbol{\theta}))^\ddagger = \mathbf{B}(\boldsymbol{\theta})^\dagger \bar{\mathbf{J}}^\top. \quad (4.23)$$

Using this and inserting (4.14) for $\mathbf{F}$ into (4.22), we can avoid transforming $\mathbf{g}$ back and forth between original and task coordinates.

Remark. This control allocation scheme effectively requires the joints to counter the torques exerted on them by the thrusters, and therefore relies on the joint motors being powerful enough to achieve that.

4.3.3 Stability analysis

Since there is no task error associated with the task velocity $\mathbf{h}_{\text{rot}}$, the completion of all tasks does not uniquely determine the state of the system. Let $\mathbf{z} = [\mathbf{z}_1^\top, \dots, \mathbf{z}_N^\top]^\top$, and let the full state of the system be described by $\boldsymbol{\chi} = [\tilde{\mathbf{p}}_{\text{cg}}^\top, \mathbf{q}^\top, \boldsymbol{\theta}^\top, \mathbf{z}^\top]^\top$, and let the complete state space be denoted $\mathcal{A}_0$. Similarly to [44], we define successive nested sets in which consecutive tasks are completed, the first of which we define as

$$\mathcal{A}_1 = \{\boldsymbol{\chi} \mid \tilde{\mathbf{p}}_{\text{cg}} = \mathbf{0}, \dot{\tilde{\mathbf{p}}}_{\text{cg}} = \mathbf{0}, \mathbf{h}_{\text{rot}} = \mathbf{0}\} \quad (4.24)$$

which is the subset of the state space in which the first task is completed. We then define the nested subsets

$$\mathcal{A}_i = \mathcal{A}_{i-1} \cap \{\boldsymbol{\chi} \mid \tilde{\mathbf{x}}_i = \mathbf{0}, \mathbf{z}_i = \mathbf{0}\} \quad (4.25)$$

for $i = 2, \dots, N$. From the expression (2.46) for $\mathbf{z}_i$ it follows that in the set $\mathcal{A}_i$ where $\tilde{\mathbf{x}}_i = \mathbf{0}$, $\dot{\tilde{\mathbf{x}}}_i = \mathbf{z}_i = \mathbf{0}$. Thus $\mathcal{A}_i$ is the subset of the state space in which all tasks up to and including task i are completed. The control objective is then to asymptotically stabilize the set $\mathcal{A}_N$ in which all tasks are completed.

Assumption 4.3. The choice of tasks is such that the dynamics (2.1) in closed-loop with the control input given by (4.14), (4.20), (2.47), (4.22) satisfy Assumption 2.1.

Remark. Assumption 4.3 is necessary in order to apply stability theorems from [99], and is a common assumption [44]. It is a reasonable assumption for the robot dynamics (2.1), but may restrict the choice of tasks.

Proposition 4.1. Consider the dynamics (2.1) in closed-loop with the control input given by (4.14), (4.20), (2.47), (4.22) and subject to Assumptions 2.2, and 4.1-4.3. Then the set $\mathcal{A}_N$ is UAS.

Proof. Since the attitude $\mathbf{q}$ and joint angles $\boldsymbol{\theta}$ evolve on compact manifolds, the set $\mathcal{A}_N$ is compact. The closed-loop dynamics of the primary task given by (2.43), (4.20), (4.18) are

$$\bar{\mathbf{M}}_1 \dot{\mathbf{z}}_1 + (\boldsymbol{\mu}_{1,1} + \boldsymbol{\delta}_{1,1} + \mathbf{K}_{D,1}) \mathbf{z}_1 + [\mathbf{K}_{P,1,p} \tilde{\mathbf{p}}_{\text{cg},r}] = \mathbf{0} \quad (4.26a)$$

$$\dot{\tilde{\mathbf{p}}}_{\text{cg}} = -\boldsymbol{\Lambda}_i \tilde{\mathbf{p}}_{\text{cg}} + (\dot{\tilde{\mathbf{p}}}_{\text{cg}} - \dot{\tilde{\mathbf{p}}}_{\text{cg},r}). \quad (4.26b)$$

Within a set $\mathcal{A}_{i-1}$, with $i \geq 2$, the closed-loop dynamics of task i given by (2.43), (2.47), (2.46) become

$$\bar{\mathbf{M}}_i \dot{\mathbf{z}}_i + (\boldsymbol{\mu}_{i,i} + \boldsymbol{\delta}_{i,i} + \mathbf{K}_{D,i}) \mathbf{z}_i + \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i = \mathbf{0}, \quad (4.27a)$$

$$\dot{\tilde{\mathbf{x}}}_i = -\boldsymbol{\Lambda}_i \tilde{\mathbf{x}}_i + \mathbf{z}_i. \quad (4.27b)$$

The analysis follows along similar lines in [44, 132], except $\mathcal{A}_N$ is now a compact set containing more than a single point. Therefore we will apply the theorems for stability of compact sets from [99], given in Section 2.2.2.

By Assumption 2.3, there exists an open neighbourhood U around the reference trajectory containing no singularities, i.e. $\mathcal{A}_N \subset U$. Within the set U , the Jacobians $\mathbf{J}^{\text{aug}}$, $\bar{\mathbf{J}}$ are invertible. At each priority level $i \geq 2$, as in [132] we choose

$$\mathbf{V}_i(\tilde{\mathbf{x}}_i, \mathbf{z}_i, \boldsymbol{\theta}) = \frac{1}{2} \mathbf{z}_i^\top \bar{\mathbf{M}}_i(\boldsymbol{\theta}) \mathbf{z}_i + \tilde{\mathbf{x}}_i^\top \mathbf{K}_{P,i} \tilde{\mathbf{x}}_i \quad (4.28)$$

and for task 1

$$V_1(\tilde{\mathbf{p}}_{cg}, \mathbf{z}_1, \boldsymbol{\theta}) = \frac{1}{2} \mathbf{z}_1^\top \bar{\mathbf{M}}_1(\boldsymbol{\theta}) \mathbf{z}_1 + \tilde{\mathbf{p}}_{cg}^\top \mathbf{K}_{P,1} \tilde{\mathbf{p}}_{cg}. \quad (4.29)$$

Let $\mathbf{y}_1 = [\tilde{\mathbf{p}}_{cg}^\top, \mathbf{z}_1^\top]^\top$, and for $i = 2, \dots, N$, let $\mathbf{y}_i = [\tilde{\mathbf{x}}_i^\top, \mathbf{z}_i^\top]^\top$. Then for $i = 1, \dots, N$ it can be shown the functions V_i (4.28), (4.29) satisfy

$$k_{1,i} \|\mathbf{y}_i\|^2 \leq V_i \leq k_{2,i} \|\mathbf{y}_i\|^2 \quad (4.30a)$$

$$\dot{V}_i \leq k_{3,i} \|\mathbf{y}_i\|^2 \quad (4.30b)$$

for $\boldsymbol{\chi} \in \mathcal{A}_{i-1} \cap U$.

The rest of the proof follows along these steps:

- 1) We first establish that $\mathcal{A}_N$ is UAS relative to $\mathcal{A}_{N-1}$.
- 2) We then establish that $\mathcal{A}_i$ is LUS near $\mathcal{A}_N$ (Definition 2.4) relative to $\mathcal{A}_{i-1}$ for each $i = 1, \dots, N-1$, using Proposition 2.1.
- 3) We then apply the following steps for each i , starting with $i = N-1$ and up to $i = 1$:
 - 3a) Using the fact that $\mathcal{A}_N$ is UAS and thus US relative to $\mathcal{A}_i$, we establish that $\mathcal{A}_i$ is t_0 -UA near $\mathcal{A}_N$ (Definition 2.5) relative to $\mathcal{A}_{i-1}$ using Corollary 2.1.
 - 3b) By applying Theorem 2.2, we conclude that $\mathcal{A}_N$ is UAS relative to $\mathcal{A}_{i-1}$.

Repeating these steps up to $i = 1$, we arrive at the conclusion that $\mathcal{A}_N$ is UAS relative to $\mathcal{A}_0$.

Step 1) From the dynamics (4.27), (4.26) we can conclude (by inserting $\mathbf{y}_i = 0$) that the sets $\mathcal{A}_i$, $i = 1, \dots, N$ are positively invariant. Therefore, for initial states $\boldsymbol{\chi}(t_0) \in \mathcal{A}_{N-1}$ the solutions remain in $\mathcal{A}_{N-1}$. Furthermore, as a consequence of the bounds (4.30), solutions starting sufficiently close to the set $\mathcal{A}_N$ will remain in U . Then V_N satisfies the bounds (4.30) along solutions starting in $\mathcal{A}_{N-1}$ and near $\mathcal{A}_N$, and we can apply the comparison lemma [78, Lemma 3.4] along the same lines as in the proof of [78, Theorem 4.10] for exponential stability of the origin. We then arrive at

$$\begin{aligned} \|\boldsymbol{\chi}(t)\|_{\mathcal{A}_N} = \|\mathbf{y}_N(t)\| &\leq \sqrt{\frac{k_{2,N}}{k_{1,N}}} \|\mathbf{y}_N(t_0)\| e^{-\frac{k_{3,N}}{2k_{2,N}}(t-t_0)} \\ &= \sqrt{\frac{k_{2,N}}{k_{1,N}}} \|\boldsymbol{\chi}(t_0)\|_{\mathcal{A}_N} e^{-\frac{k_{3,N}}{2k_{2,N}}(t-t_0)} \end{aligned} \quad (4.31)$$

From (4.31) and Definition 2.6 of relative stability properties, we can conclude that $\mathcal{A}_N$ is UAS relative to $\mathcal{A}_{N-1}$.

Step 2) Within the set $\mathcal{A}_{i-1}$, the function V_i satisfies (4.30), and from (4.30a) we have

$$k_{1,i} \|\boldsymbol{\chi}\|_{\mathcal{A}_i}^2 = k_{1,i} \|\mathbf{y}_i\|^2 \leq V_i \leq k_{2,i} \|\mathbf{y}_i\|^2 \leq k_{2,i} \sum_{j=i}^N \|\mathbf{y}_j\|^2 = k_{2,i} \|\boldsymbol{\chi}\|_{\mathcal{A}_N}^2 \quad (4.32)$$

for $\boldsymbol{\chi} \in \mathcal{A}_{i-1}$, where $\|\cdot\|_{\mathcal{A}_j}$ denotes the distance to $\mathcal{A}_j$. Hence by Proposition 2.1, each $\mathcal{A}_i$ is LUS near $\mathcal{A}_N$ relative to $\mathcal{A}_{i-1}$ for $i = 1, \dots, N-1$.

Step 3a) Starting with $i = N - 1$, $\mathcal{A}_N$ is UAS and thus US relative to $\mathcal{A}_i$. Again, within the set $\mathcal{A}_{i-1}$, the function V_i satisfies (4.30), and thus by Corollary 2.1, we conclude that $\mathcal{A}_i$ is t_0 -UA near $\mathcal{A}_N$ relative to $\mathcal{A}_i$.

Step 3b) Since $\mathcal{A}_N$ is UAS relative to $\mathcal{A}_i$, and $\mathcal{A}_i$ is LUS near $\mathcal{A}_N$ and t_0 -UA near $\mathcal{A}_N$, both relative to $\mathcal{A}_{i-1}$, by Theorem 2.2 $\mathcal{A}_N$ is UAS relative to $\mathcal{A}_{i-1}$. $\square$

Optional inclusion of added mass approximation

If some knowledge of the added mass effects is available, Assumption 4.2 could be replaced with:

Assumption 4.2'. The added mass effects acting on each link can be modelled such that the upper left 3×3 -block of each link's respective inertia matrix about a frame with origin at the CG of the link can be written as $(m_i + m_{a,i})\mathbf{I}_3$, where m_i is the rigid body mass of the link, and $m_{a,i}$ is an added mass coefficient.

This assumption can be satisfied by links with uniformly distributed mass, so that the CG is located at the centroid of the link, and with shapes satisfying sufficient symmetry conditions. Such shapes include spheres or cubes [138]. While those are not typical shapes for links of a manipulator systems, for some manipulators, e.g. with short, cylindrical links, it might be a useful approximation in order to include and thus compensate added mass effects to some extent. Similarly as when neglecting the added mass entirely, deviation from the assumption will result in a disturbance from unmodeled added mass effects which is not explicitly compensated for by the controller.

4.4 Simulation results

The proposed control method is simulated on an AIAUV model like the one in Section 2.1.2. In order to satisfy Assumption 4.1, the simulated AIAUV is augmented with two additional thrusters on its center link to ensure that it is always actuated in roll. Consequently, the simulated AIAUV is equipped with $m = 9$ thrusters in this chapter. These are illustrated in Figure 4.2.

We choose $N = 3$ tasks as follows: the primary task is to control the position of the CG and total rotational momentum, as described in Section 4.3. As task 2 we choose the pose of the end-effector, with task dimension $m_2 = 6$. Finally, as task 3 we choose to point the opposite end, i.e. the base link of the AIAUV, by controlling its pitch and yaw, with task dimension $m_3 = 2$. The sum of dimensions of the chosen tasks is then equal to the number of available DOFs, in accordance with Assumption 2.2. The control gains are chosen to be $\mathbf{\Lambda}_{1,p} = \mathbf{K}_{P,1,p} = \mathbf{K}_{D,1,h} = \mathbf{I}_3$, $\mathbf{K}_{D,1,p} = 5\mathbf{I}_3$, $\mathbf{\Lambda}_2 = \mathbf{K}_{D,2} = \mathbf{I}_6$, $\mathbf{K}_{P,2} = \begin{bmatrix} 5\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & 10\mathbf{I}_3 \end{bmatrix}$, $\mathbf{\Lambda}_3 = 0.1\mathbf{I}_2$, $\mathbf{K}_{P,3} = 7.5\mathbf{I}_2$ and $\mathbf{K}_{D,3} = 5\mathbf{I}_2$. The simulated dynamics include added mass effects, which were not considered in the derivation of the decoupling coordinate transformation. The control input is calculated based only on the rigid body mass, which allows us to test the robustness of the proposed control approach against these effects.

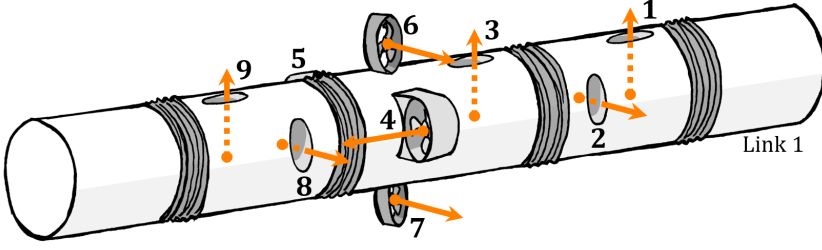


Figure 4.2: Illustration of thruster placement and directions for the simulated AIAUV. The placement of thruster no. 5 mirrors that of no. 4.

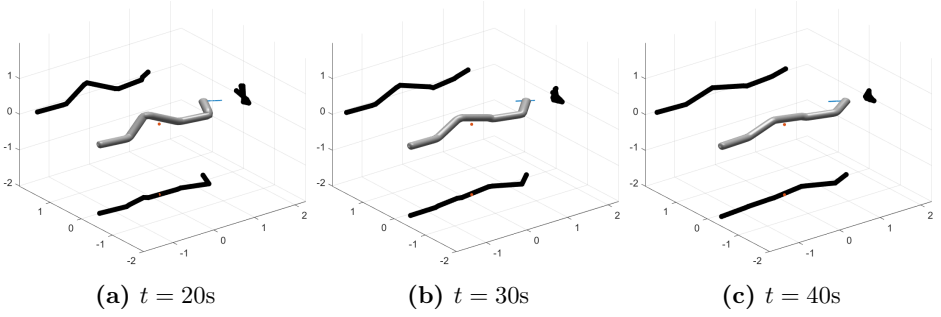


Figure 4.3: Simulation snapshots, with CG in red and the line traced out by the end-effector trajectory in blue.

Figure 4.3 show snapshots from the simulation, where it can be seen that in order to track the desired end-effector trajectory without disturbing the position of the CG, the AIAUV has to contract its body. The reference trajectories and the task trajectories are shown in Figure 4.4. The joint angles, torques and thruster forces are shown in Figure 4.5, and the norms of the tracking errors are shown in Figure 4.6. For the second task, the error variables $\tilde{x}_2$, s_2 are split into translational and rotational parts before their norms are taken.

As can be seen in Figures 4.4 and 4.6, all objectives are tracked well, even when the conditions of Proposition 4.1 are not fully met, due to the presence of added mass effects. The position of the CG converges to its desired value, after which it remains within less than 1 cm of its reference position despite the significant movement of the end-effector. Figure 4.6 shows that all the tracking errors are bounded. The lowest-priority task has the largest error, which is to be expected, as errors in higher-priority tasks give top-down disturbances in the task hierarchy. On the other hand, we see in Figure 4.4a that there are some small deviations in the position of the CG from its desired position, despite this being the task with the highest priority. While present, these deviations are less than 1 cm, demonstrating that in the simulated scenario the positioning of the CG works well, despite the presence of unaccounted for added mass effects. The added mass effects, which were not taken into account in the decoupling coordinate transform in the control input, introduce a disturbance to each individual task, as well as a coupling between the tasks, causing

bottom-up disturbances in the task hierarchy. Taking a simplified view of the interconnections between tasks as time-varying signals when evaluated along trajectories of the system, the individual task dynamics (4.27) can be viewed as exponentially stable subsystems subject to a disturbance. In the presence of an additional disturbance, such as caused by added mass effects, higher-priority tasks may not converge entirely to their references and these top-down disturbances will not necessarily vanish. However, the exponential stability of each subsystem provides some robustness against non-vanishing bounded disturbances [78, Lemma 9.2]. In order to further improve performance in the presence of uncompensated disturbances, a robust control method could be applied, as in e.g. [45].

The workspace available to the end-effector when using the GJM, i.e. the workspace which is reachable without shifting the CG of the system, is smaller than the workspace reachable by the same manipulator if its base were fixed. In the simulated scenario, the desired end-effector trajectory is to follow a straight line in the xy -plane back and forth. In order to reach points in the interior its workspace without shifting the position of the CG, the AIAUV must contract its body using all its joints. This may require large joint angles, as can be seen in Figure 4.5a. The workspace in which the end-effector can realistically operate is therefore even smaller. The reduced workspace size, when taken together with the requirements posed by Assumptions 2.2 and 2.3 that all DOFs must be employed and references must be compatible, requires a thorough planning phase beforehand to ensure the coordination between repositioning the CG and the motion of the end-effector.

Figure 4.5 shows that the applied thruster forces and joint torques stay within reasonable values throughout the simulation, demonstrating that the proposed control allocation scheme is feasible. It should be noted however, that even though the motivation for the proposed control method was to reduce the use of thrusters, the thrusters are active at all times. This is due to the cancellation of the hydrodynamic damping and hydrostatic terms. The hydrodynamic damping terms would otherwise introduce a coupling between tasks and cause an undesired bottom-up disturbance in the task hierarchy. Cancelling or compensating for this disturbance is necessary in an underwater setting regardless of the choice of tasks. However, in Figure 4.5c it can be seen that the largest thrust is used to produce a roll moment, which is more consistent with compensation of hydrostatic terms. Producing a roll moment requires large forces due to the small arm of the thrusters capable of producing such a moment. The need for a roll moment suggests that the given task trajectories may require the vehicle to be in configurations which are challenging for a ballasted vehicle, like the one simulated.

The ability of the control method proposed in this chapter to accurately control the position of the CG while performing other tasks shows great promise, and further simulation studies should be performed to compare its performance and required control effort against other choices of tasks. Due to the requirement posed by Assumption 2.2 that tasks must fully occupy all DOFs, it may be difficult to find different sets of tasks which still give a fair comparison.

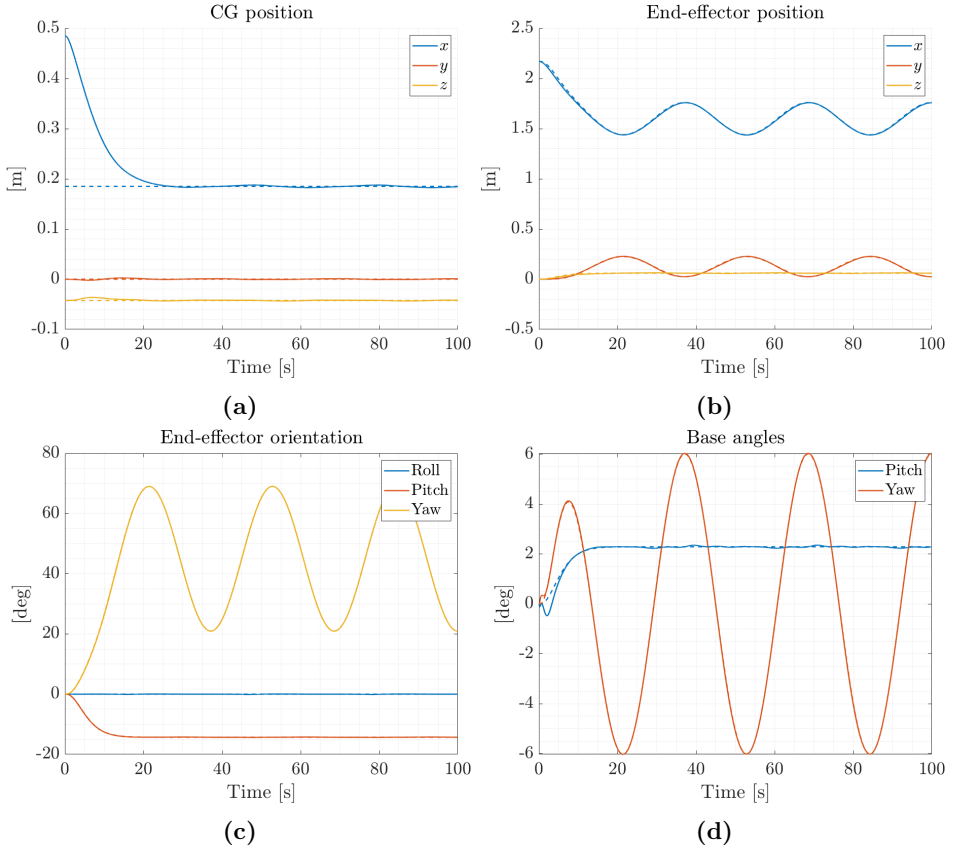


Figure 4.4: Task trajectories, with their corresponding reference trajectories as dashed lines

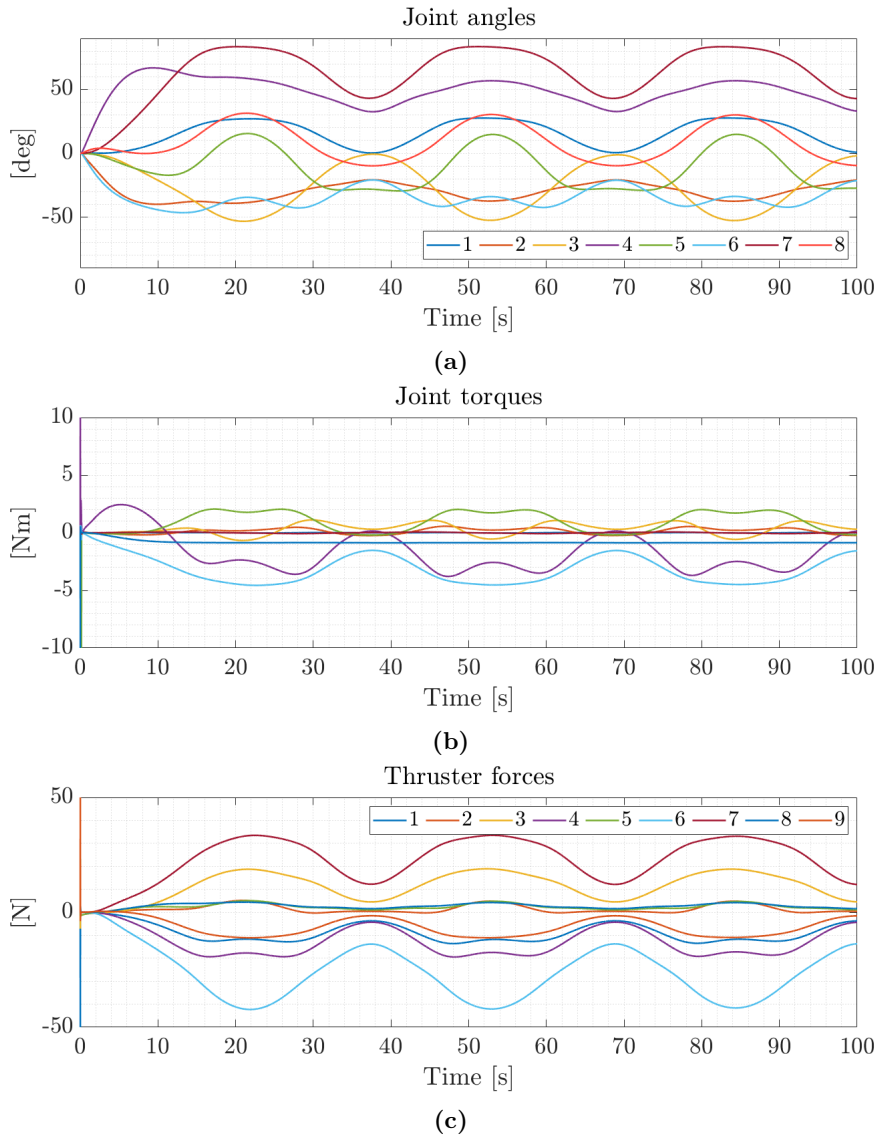
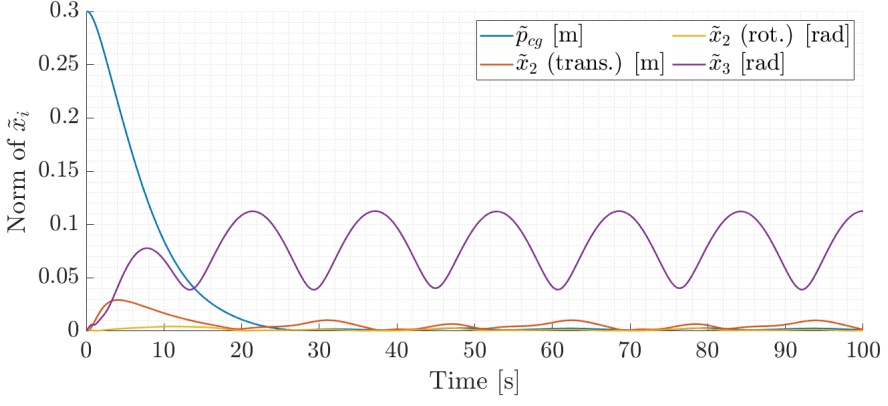
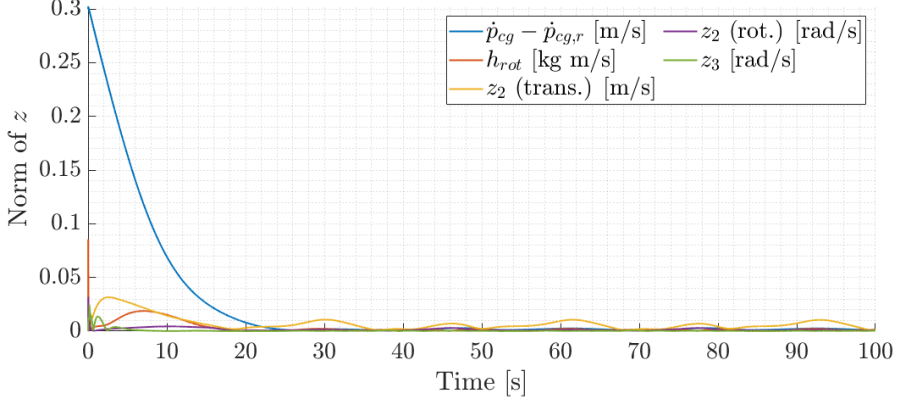


Figure 4.5: Joint angles and applied torques and thruster forces



(a) Task tracking errors



(b) Task velocity tracking errors

Figure 4.6: Error norms

4.5 Chapter summary

In this chapter, the GJM was derived as a special case of a prioritized Jacobian matrix which is projected into the null-space of total momentum or related tasks such as position of the CG. This insight was used in a proposed control method in which the motion of the CG was chosen as the first task in a prioritized task hierarchy comprised of an arbitrary number of additional tracking tasks. A control allocation scheme using a (1,2,3)-pseudoinverse was proposed to ensure that the thrusters are used only for moving the CG, while the joint torques are used to accomplish precise tracking of other task trajectories. The set in which all task tracking errors are zero was shown to be uniformly asymptotically stable, and the proposed control method is demonstrated to work well in simulation.

Chapter 5

Collision Avoidance using Mixed H_2/H_∞ Control

This chapter considers the problem of regulating the joints of an AIAUV to desired setpoints in the presence of parameter uncertainties, while simultaneously avoiding collisions. In order to obtain a trade-off between optimality and robustness, we use the method of H_2/H_∞ control, posed as a differential game in which a control input and a disturbance are the players. We find a solution to a set of matrix equations required to apply a general form of approximate solution to the differential game problem to a system with double integrator dynamics. Combined with feedback linearization, the resulting optimal control input is applied to the joints of an AIAUV. The joints are controlled to reach desired setpoints while avoiding self-collisions, with simulations demonstrating the performance.

Contributions of this chapter

The first contribution of this chapter is finding an algebraic $\bar{P}$ solution for the mixed H_2/H_∞ control problem posed as a differential game for systems with double integrator dynamics. Then, a control method for the joints of an AIAUV is proposed for regulation to desired setpoints while avoiding joint limits and self-collisions, in a manner which is robust to uncertainty in the parameters of the system dynamics. This is done by combining the optimal control input for double integrator dynamics with a feedback linearizing controller and suitable cost functions. The complete control scheme is applied to an AIAUV in simulations.

Publications

This chapter is based on [158].

Organization of this chapter

This chapter is organized as follows: first, an introduction is given Section 5.1. The method for constructing solutions for mixed H_2/H_∞ control problems posed as a differential game for a general system from [107] is recalled in Section 5.2, after which the specific solution for double integrator dynamics is found in Section 5.3.

In Section 5.4 the feedback linearizing input for the joints, and the objective used for collision avoidance are presented. Simulations illustrating the results are shown and discussed in Section 5.5, and a summary of the chapter is given in Section 5.6.

5.1 Introduction

Mixed H_2/H_∞ control addresses the problem of optimal control, in which multiple objectives are gathered into one cost function to be minimized, in the presence of a disturbance acting on the system. This enables combining both robustness and multiple control objectives, such as moving the joints while simultaneously avoiding collisions. Since all objectives are gathered into a common cost function, this method only allows for soft prioritizing of multiple objectives, unlike the hard prioritization achieved by the use of null-space projections used in Chapters 3 and 4, or by [41, 104] for collision avoidance.

Mixed H_2/H_∞ control can be viewed as differential game, in which a control input and an exogenous disturbance are the players [97, 107]. While [97] addresses the problem for linear systems, in [107] it is addressed for a more general class of nonlinear systems, where both system dynamics and control objectives may be nonlinear. The framework of nonlinear differential games is also well-suited for control of multiagent systems and e.g. collision avoidance between agents [108, 109]. The solution to a differential game problem are control inputs, or strategies for each participant which are optimal given the strategies of the other players, which constitutes what is called a Nash equilibrium [112]. Such solutions are in general very difficult to find for nonlinear games. A general expression for approximate solutions to differential game problems was presented in [106]. These solutions are strategies such that each player may only achieve a small, bounded gain by changing their strategy. These strategies require the solution to a set of matrix equations for the system dynamics to which they are to be applied, which may be difficult to solve. The agents considered in [108, 109] are described by single integrator dynamics, or their dynamics are feedback linearized to a single integrator. An explicit solution for a system with more complex dynamics, specifically a manipulator not subject to any damping or gravitational forces, was found in [136], but only for either optimal control or H_∞ -control. Mixed H_2/H_∞ control for robotic manipulators formulated as a differential game was also previously considered in e.g. [32] for tracking joint angle trajectories, but only with a quadratic optimization objective with constant weights.

The dynamics of an AIAUV are complicated and include hydrodynamic effects, which are difficult to model and estimate. The solution to a mixed H_2/H_∞ control problem is designed to be robust against disturbances, but is best suited to systems with relatively simple dynamics for which the matrices required to apply the approximate solution to the differential game problem can be found [106, 107]. On the other hand, the method of feedback linearization can be used to simplify complex nonlinear dynamics into simpler, linear dynamics through feedback, but requires exact cancellation of the nonlinear dynamics [78]. Otherwise, imperfect cancellation of these dynamics will lead to a disturbance from the uncompensated dynamic effects. By combining the two methods, the mixed H_2/H_∞ control problem can be solved

for the simplified linear dynamics, and applied together with a feedback linearizing input based on an estimate of the original dynamics. Since the mixed H_2/H_∞ control input is designed to provide robustness, it can be used to mitigate the disturbance from uncompensated dynamics. In order to apply mixed H_2/H_∞ control solved as a differential game to an AIAUV, we feedback linearize its dynamics to a double integrator. For these simplified dynamics, we find the solution required to apply the control input designed in [106, 107].

The approach to collision avoidance from [108, 109] is somewhat similar to artificial potential fields, designed to push a robot or parts of it away from obstacles and towards a goal [79]. This is a local approach and vulnerable to the presence of local optima or equilibria. This issue has been addressed in [127], for scenarios in which the location of obstacles in the environment is known beforehand. Local equilibria such as e.g. deadlocks are also avoided in [108, 109], thanks to an extended state required to construct the approximate solution. This extension also acts similarly to a virtual reference which helps guide the agents out of deadlocks.

5.2 Mixed H_2/H_∞ control as a differential game

This section introduces the method for constructing approximate solutions to the differential game formulation of the mixed H_2/H_∞ control problem provided in [107].

The systems considered in [107] are general nonlinear systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}_1(\mathbf{x})\mathbf{w} + \mathbf{g}_2(\mathbf{x})\mathbf{u} \quad (5.1)$$

with state $\mathbf{x} \in \mathbb{R}^m$, a disturbance or exogenous input $\mathbf{w} \in \mathbb{R}^{m_1}$, and a control input $\mathbf{u} \in \mathbb{R}^{m_2}$. The functions $\mathbf{f}(\mathbf{x})$, $\mathbf{g}_i(\mathbf{x})$ with $i = 1, 2$ are continuous. The origin is assumed to be an equilibrium of the system (5.1) and $\mathbf{f}(\mathbf{x})$ sufficiently smooth, such that there exists a mapping $\mathbf{F}(\mathbf{x})$ with $\mathbf{f}(\mathbf{x}) = \mathbf{F}(\mathbf{x})\mathbf{x}$.

The system (5.1) has the associated performance variables

$$\mathbf{z}_1 = \mathbf{h}_1(\mathbf{x}) + \mathbf{k}_1(\mathbf{x})\mathbf{u}, \quad \mathbf{z}_2 = \mathbf{h}_2(\mathbf{x}) + \mathbf{k}_2(\mathbf{x})\mathbf{u} \quad (5.2)$$

with $\mathbf{z}_1 \in \mathbb{R}^{p_1}$, $\mathbf{z}_2 \in \mathbb{R}^{p_2}$ and where $\mathbf{h}_i(\mathbf{x})$, $\mathbf{k}_i(\mathbf{x})$ with $i = 1, 2$ are continuous in $\mathbf{x}$. The performance variables $\mathbf{z}_1$ and $\mathbf{z}_2$ will be associated with the H_∞ and the H_2 criteria in the mixed H_2/H_∞ control problem, respectively. The mappings in (5.2) are assumed to satisfy $\mathbf{h}_i(\mathbf{0}) = \mathbf{0}$, $\mathbf{h}_i(\mathbf{x})^\top \mathbf{k}_i(\mathbf{x}) = \mathbf{0}$ and $\mathbf{k}_i(\mathbf{x})^\top \mathbf{k}_i(\mathbf{x}) = \mathbf{I}_{m_2}$ for $i = 1, 2$. From the first of these conditions it follows that we can write

$$\mathbf{h}_i(\mathbf{x})^\top \mathbf{h}_i(\mathbf{x}) = \mathbf{x}^\top \mathbf{Q}_i(\mathbf{x})\mathbf{x} \quad (5.3)$$

where $\mathbf{Q}_i(\mathbf{x}) \geq 0$ for $i = 1, 2$.

The mixed H_2/H_∞ control problem can be stated as follows:

Problem 5.1. Given the system (5.1), determine a feedback control law $\mathbf{u}^*$ such that

- i) The origin is a (locally) asymptotically stable equilibrium of the system when $\mathbf{w}(t) = \mathbf{w}^*$, $\mathbf{w}^*$ being the worst-case disturbance, with region of attraction including a non-empty neighbourhood $\bar{\Omega}$ containing the origin.

- ii) For every $\mathbf{w} \in \mathcal{L}_2$ such that the trajectories of the system remain in $\bar{\Omega}$, the $\mathcal{L}_2$ -gain from $\mathbf{w}$ to $\mathbf{z}_1$ is less than some $\gamma > 0$, i.e.

$$\int_0^T \|\mathbf{z}_1\|^2 dt \leq \gamma^2 \int_0^T \|\mathbf{w}\|^2 dt \quad (5.4)$$

- iii) The control input $\mathbf{u}^*$ regulates the state $\mathbf{x}(t)$ in such a way as to minimize the output energy with respect to the output $\mathbf{z}_2$ when the worst case disturbance $\mathbf{w}^*$ is applied to the system, i.e. $\mathbf{u}^*$ minimizes

$$J_2(\mathbf{u}, \mathbf{w}) = \frac{1}{2} \int_0^T \mathbf{z}_2^\top \mathbf{z}_2 dt = \frac{1}{2} \int_0^T \mathbf{x}^\top \mathbf{Q}_2(\mathbf{x}) \mathbf{x} + \mathbf{u}^\top \mathbf{u} dt \quad (5.5)$$

when $\mathbf{w} = \mathbf{w}^*$.

A function $\boldsymbol{\alpha} : [0, \infty) \rightarrow \mathbb{R}^n$ is said to belong to the space $\mathcal{L}_2$ if it satisfies $\|\boldsymbol{\alpha}\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty \boldsymbol{\alpha}(t)^\top \boldsymbol{\alpha}(t) dt} < \infty$.

We consider the infinite horizon case, i.e. when $T \rightarrow \infty$. Problem 5.1 can be formulated as two-player differential game (see [97]). To this end, let

$$\begin{aligned} J_1(\mathbf{u}, \mathbf{w}) &= \frac{1}{2} \int_0^\infty -\mathbf{z}_1^\top \mathbf{z}_1 + \gamma^2 \mathbf{w}^\top \mathbf{w} dt \\ &= \frac{1}{2} \int_0^\infty -\mathbf{x}^\top \mathbf{Q}_1(\mathbf{x}) \mathbf{x} - \mathbf{u}^\top \mathbf{u} + \gamma^2 \mathbf{w}^\top \mathbf{w} dt . \end{aligned} \quad (5.6)$$

Problem 5.2. Consider the system (5.1). Determine a set of admissible feedback strategies $\mathbf{u}^*$, $\mathbf{w}^*$ which satisfy the Nash equilibrium inequalities

$$J_1(\mathbf{u}^*, \mathbf{w}^*) \leq J_1(\mathbf{u}^*, \mathbf{w}), \quad J_2(\mathbf{u}^*, \mathbf{w}^*) \leq J_2(\mathbf{u}, \mathbf{w}^*) \quad (5.7)$$

for all admissible pairs $(\mathbf{u}, \mathbf{w}^*)$ and $(\mathbf{u}^*, \mathbf{w})$.

A pair of feedback strategies $(\mathbf{u}, \mathbf{w})$ is said to be admissible if the origin of the system (5.1) in closed loop with $(\mathbf{u}, \mathbf{w})$ is a (locally) asymptotically stable equilibrium point.

Note that

$$J_1(\mathbf{u}, \mathbf{w}) \geq 0 \quad \Rightarrow \quad \int_0^T \|\mathbf{z}_1\|^2 dt \leq \gamma^2 \int_0^T \|\mathbf{w}\|^2 dt . \quad (5.8)$$

The factor $\gamma > 0$ in (5.6) is referred to as a disturbance attenuation level. When $J_1(\mathbf{u}, \mathbf{w}) \geq 0$, γ is the $\mathcal{L}_2$ -gain from the disturbance $\mathbf{w}$ to $\mathbf{z}_1$.

Solving this differential game requires the solution of a set of partial differential equations which are in general difficult to solve for nonlinear systems. Instead of finding the exact solution, a means of constructing approximate solutions was developed in [106], and applied to the mixed H_2/H_∞ control problem in [107]. A dynamic extension $\boldsymbol{\xi}(t) \in \mathbb{R}^m$ is introduced, along with the extended value functions

$$\begin{aligned} V_1(\mathbf{x}, \boldsymbol{\xi}) &= \frac{1}{2} \mathbf{x}^\top \mathbf{P}_1(\boldsymbol{\xi}) \mathbf{x} - \frac{1}{2} (\mathbf{x} - \boldsymbol{\xi})^\top \mathbf{R}_1(\mathbf{x} - \boldsymbol{\xi}) , \\ V_2(\mathbf{x}, \boldsymbol{\xi}) &= \frac{1}{2} \mathbf{x}^\top \mathbf{P}_2(\boldsymbol{\xi}) \mathbf{x} + \frac{1}{2} (\mathbf{x} - \boldsymbol{\xi})^\top \mathbf{R}_2(\mathbf{x} - \boldsymbol{\xi}) \end{aligned} \quad (5.9)$$

where $\mathbf{R}_i = \mathbf{R}_i^\top > 0$, and $\mathbf{P}_1(\mathbf{x}) \leq 0$, $\mathbf{P}_2(\mathbf{x}) \geq 0$ are so-called algebraic $\bar{P}$ solutions for the system (5.1), (5.2), to be introduced in Section 5.3.1. Furthermore, the system (5.1) with output $y = \mathbf{x}^\top \mathbf{Q}_1(\mathbf{x})\mathbf{x} + \mathbf{x}^\top \mathbf{Q}_2(\mathbf{x})\mathbf{x}$ must satisfy the following assumption.

Assumption 5.1. The pair $\{\mathbf{f}, y\}$ and the pair $\{\mathbf{f} - \frac{1}{\gamma^2} \mathbf{g}_1(\mathbf{x})\mathbf{g}_1(\mathbf{x})^\top \left(\frac{\partial V_1}{\partial \mathbf{x}}\right)^\top, y\}$ are both zero-state detectable.

An approximate solution of Problem 5.2 is provided in the following statement (see [106, 107] for details).

Theorem 5.1 ([107], Theorem 1). *Consider the system (5.1), the cost functionals (5.6), (5.5) and the resulting non-zero-sum differential game in Problem 5.2. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be such that $\mathbf{R}_2 > \mathbf{R}_1$ and $\mathbf{R}_i(\mathbf{R}_1 + \mathbf{R}_2) + (\mathbf{R}_1 + \mathbf{R}_2)\mathbf{R}_i > 0$, for $i = 1, 2$. There exists a neighbourhood Ω , containing the origin, and $\bar{\kappa} > 0$ such that for all $\kappa \geq \bar{\kappa}$ the inequalities*

$$\begin{aligned} \mathcal{HJ}\mathcal{I}_1 = & -\frac{1}{2} \mathbf{h}_1(\mathbf{x})^\top \mathbf{h}_1(\mathbf{x}) - \frac{1}{2} \frac{\partial V_2}{\partial \mathbf{x}} \mathbf{g}_2(\mathbf{x}) \mathbf{g}_2(\mathbf{x})^\top \left(\frac{\partial V_2}{\partial \mathbf{x}}\right)^\top \\ & - \frac{1}{2\gamma^2} \frac{\partial V_1}{\partial \mathbf{x}} \mathbf{g}_1(\mathbf{x}) \mathbf{g}_1(\mathbf{x})^\top \left(\frac{\partial V_1}{\partial \mathbf{x}}\right)^\top + \frac{\partial V_1}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) \\ & - \frac{1}{2} \frac{\partial V_1}{\partial \mathbf{x}} \mathbf{g}_2(\mathbf{x}) \mathbf{g}_2(\mathbf{x})^\top \left(\frac{\partial V_2}{\partial \mathbf{x}}\right)^\top + \frac{\partial V_1}{\partial \boldsymbol{\xi}} \dot{\boldsymbol{\xi}} \geq 0, \end{aligned} \quad (5.10a)$$

$$\begin{aligned} \mathcal{HJ}\mathcal{I}_2 = & \frac{1}{2} \mathbf{h}_2(\mathbf{x})^\top \mathbf{h}_2(\mathbf{x}) - \frac{1}{2} \frac{\partial V_2}{\partial \mathbf{x}} \mathbf{g}_2(\mathbf{x}) \mathbf{g}_2(\mathbf{x})^\top \left(\frac{\partial V_2}{\partial \mathbf{x}}\right)^\top \\ & + \frac{\partial V_2}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) - \frac{1}{2} \frac{\partial V_2}{\partial \mathbf{x}} \mathbf{g}_2(\mathbf{x}) \mathbf{g}_2(\mathbf{x})^\top \left(\frac{\partial V_1}{\partial \mathbf{x}}\right)^\top + \frac{\partial V_2}{\partial \boldsymbol{\xi}} \dot{\boldsymbol{\xi}} \leq 0 \end{aligned} \quad (5.10b)$$

with

$$\dot{\boldsymbol{\xi}} = -\kappa \left(\frac{\partial V_2}{\partial \boldsymbol{\xi}} - \frac{\partial V_1}{\partial \boldsymbol{\xi}} \right)^\top, \quad (5.11)$$

are satisfied for all $(\mathbf{x}, \boldsymbol{\xi}) \in \Omega$. Suppose Assumption 5.1 is satisfied with V_1, V_2 given by (5.9). Then, the dynamic feedback strategy

$$\mathbf{u}^* = -\mathbf{g}_2(\mathbf{x})^\top \left(\frac{\partial V_2}{\partial \mathbf{x}}\right)^\top, \quad \mathbf{w}^* = -\frac{1}{\gamma^2} \mathbf{g}_1(\mathbf{x})^\top \left(\frac{\partial V_1}{\partial \mathbf{x}}\right)^\top \quad (5.12)$$

is such that the closed-loop system (5.1), (5.11), (5.12) is (locally) asymptotically stable.

The optimal input and worst case disturbance given by (5.12) are the Nash equilibrium strategies of a modified differential game with modified cost functionals $\tilde{J}_1(\mathbf{u}, \mathbf{w}) \leq J_1(\mathbf{u}, \mathbf{w})$ and $\tilde{J}_2(\mathbf{u}, \mathbf{w}) \geq J_2(\mathbf{u}, \mathbf{w})$ (see [97, 107] for details).

5.3 Algebraic $\bar{P}$ solution for double integrator dynamics

In this section, the solution required to apply the result from Section 5.2 to a system described by double integrator dynamics is derived. While the dynamics themselves are linear, the performance variables will be nonlinear. Consequently, solutions for linear systems such as those in e.g. [97] cannot be applied.

Consider now a system described by double integrator dynamics, influenced by a disturbance at the acceleration level, i.e.

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{0}_{n \times n} & \mathbf{I}_n \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{0}_{n \times n} \\ \mathbf{I}_n \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0}_{n \times n} \\ \mathbf{I}_n \end{bmatrix} \mathbf{u} \quad (5.13)$$

with $\mathbf{x} \in \mathbb{R}^{2n}$ and $\mathbf{w}, \mathbf{u} \in \mathbb{R}^n$.

The system (5.13) has associated performance variables (5.2) as defined previously. The matrices $\mathbf{Q}_1(\mathbf{x}), \mathbf{Q}_2(\mathbf{x})$ are now assumed to satisfy the following:

Assumption 5.2. The matrices $\mathbf{Q}_1(\mathbf{x})$ and $\mathbf{Q}_2(\mathbf{x})$ can be partitioned into $n \times n$ -blocks

$$\mathbf{Q}_i(\mathbf{x}) = \begin{bmatrix} \mathbf{Q}_{i11}(\mathbf{x}) & \mathbf{Q}_{i12}(\mathbf{x}) \\ \mathbf{Q}_{i12}(\mathbf{x})^\top & \mathbf{Q}_{i22}(\mathbf{x}) \end{bmatrix} \quad (5.14)$$

where the blocks $\mathbf{Q}_{i12}(\mathbf{x})$ satisfy

$$\mathbf{Q}_{i12}(\mathbf{x})^\top \mathbf{Q}_{i12}(\mathbf{x}) < b_1 b_2 \mathbf{I}_n \quad \forall \mathbf{x} \quad (5.15)$$

for $i = 1, 2$ and some constants $b_1, b_2 > 0$.

5.3.1 Algebraic $\bar{P}$ matrix solution

An $\mathcal{X}$ -algebraic $\bar{P}$ solution for the general, nonlinear system (5.1), (5.2) for the mixed H_2/H_∞ control problem is defined in [107] as a set of matrix-valued $\mathcal{C}^1$ functions $\mathbf{P}_1(\mathbf{x}), \mathbf{P}_2(\mathbf{x}) \in \mathbb{R}^{m \times m}$, with $\mathbf{P}_i(\mathbf{x}) = \mathbf{P}_i(\mathbf{x})^\top$, $i = 1, 2$, and such that for all $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^m$ they satisfy

$$\begin{aligned} & -\mathbf{Q}_1(\mathbf{x}) - \mathbf{P}_2(\mathbf{x})\mathbf{g}_2(\mathbf{x})\mathbf{g}_2(\mathbf{x})^\top \mathbf{P}_2(\mathbf{x}) \\ & -\frac{1}{\gamma^2} \mathbf{P}_1(\mathbf{x})\mathbf{g}_1(\mathbf{x})\mathbf{g}_1(\mathbf{x})^\top \mathbf{P}_1(\mathbf{x}) + \mathbf{P}_1(\mathbf{x})\mathbf{F}(\mathbf{x}) + \mathbf{F}(\mathbf{x})^\top \mathbf{P}_1(\mathbf{x}) \\ & -\mathbf{P}_1(\mathbf{x})\mathbf{g}_2(\mathbf{x})\mathbf{g}_2(\mathbf{x})^\top \mathbf{P}_2(\mathbf{x}) - \mathbf{P}_2(\mathbf{x})\mathbf{g}_1(\mathbf{x})\mathbf{g}_1(\mathbf{x})^\top \mathbf{P}_1(\mathbf{x}) - \Sigma_1(\mathbf{x}) = \mathbf{0}, \end{aligned} \quad (5.16a)$$

$$\begin{aligned} & \mathbf{Q}_2(\mathbf{x}) - \mathbf{P}_2(\mathbf{x})\mathbf{g}_2(\mathbf{x})\mathbf{g}_2(\mathbf{x})^\top \mathbf{P}_2(\mathbf{x}) \\ & + \mathbf{P}_2(\mathbf{x})\mathbf{F}(\mathbf{x}) + \mathbf{F}(\mathbf{x})^\top \mathbf{P}_2(\mathbf{x}) - \frac{1}{\gamma^2} \mathbf{P}_2(\mathbf{x})\mathbf{g}_1(\mathbf{x})\mathbf{g}_1(\mathbf{x})^\top \mathbf{P}_1(\mathbf{x}) \\ & -\frac{1}{\gamma^2} \mathbf{P}_1(\mathbf{x})\mathbf{g}_1(\mathbf{x})\mathbf{g}_1(\mathbf{x})^\top \mathbf{P}_2(\mathbf{x}) + \Sigma_2(\mathbf{x}) = \mathbf{0}, \end{aligned} \quad (5.16b)$$

where $\Sigma_i(\mathbf{x}) \in \mathbb{R}^{m \times m}$ are symmetric, positive semidefinite matrix-valued functions satisfying $\Sigma_i(\mathbf{0}) > 0$ for $i = 1, 2$. In addition, the matrices $\mathbf{P}_1(\mathbf{x}), \mathbf{P}_2(\mathbf{x})$ must satisfy $\bar{\mathbf{P}}_2 - \bar{\mathbf{P}}_1 > 0$, where $\bar{\mathbf{P}}_i = \mathbf{P}_i(\mathbf{0})$, $i = 1, 2$, solve a set of coupled Riccati equations for the system (5.1), (5.2) linearized about the origin. Because the dynamics of the system (5.13) are linear, the linearized coupled Riccati equations coincide with the equations (5.16) evaluated at $\mathbf{x} = \mathbf{0}$. The matrices $\mathbf{P}_1(\mathbf{x})$ and $\mathbf{P}_2(\mathbf{x})$ are called algebraic $\bar{P}$ solutions if the equations (5.16) hold for all $\mathbf{x} \in \mathbb{R}^{2n}$, i.e. if $\mathcal{X} = \mathbb{R}^{2n}$.

5.3.2 Solving for double integrator dynamics

The dependency of $\mathbf{P}_i$, $\mathbf{Q}_i$, $\mathbf{\Sigma}_i$ on $\mathbf{x}$ will now be omitted for brevity. Let

$$\mathbf{P}_i = \begin{bmatrix} \mathbf{P}_{i11} & \mathbf{P}_{i12} \\ \mathbf{P}_{i12}^\top & \mathbf{P}_{i22} \end{bmatrix}, \quad \mathbf{\Sigma}_i = \begin{bmatrix} \mathbf{\Sigma}_{i11} & \mathbf{\Sigma}_{i12} \\ \mathbf{\Sigma}_{i12}^\top & \mathbf{\Sigma}_{i22} \end{bmatrix}. \quad (5.17)$$

Substituting (5.13) for the double integrator dynamics and (5.2) for the performance variables into (5.16) yields a set of six equations:

$$-\mathbf{Q}_{111} - \mathbf{P}_{212}\mathbf{P}_{212}^\top - \frac{1}{\gamma^2}\mathbf{P}_{112}\mathbf{P}_{112}^\top - \mathbf{P}_{112}\mathbf{P}_{212}^\top - \mathbf{P}_{212}\mathbf{P}_{112}^\top - \mathbf{\Sigma}_{111} = \mathbf{0}, \quad (5.18a)$$

$$-\mathbf{Q}_{112} - \mathbf{P}_{212}\mathbf{P}_{222} - \frac{1}{\gamma^2}\mathbf{P}_{112}\mathbf{P}_{122} \quad (5.18b)$$

$$+ \mathbf{P}_{111} - \mathbf{P}_{112}\mathbf{P}_{222} - \mathbf{P}_{212}\mathbf{P}_{122} - \mathbf{\Sigma}_{112} = \mathbf{0},$$

$$-\mathbf{Q}_{122} - \mathbf{P}_{222}^2 - \frac{1}{\gamma^2}\mathbf{P}_{122}^2 + \mathbf{P}_{112}^\top \quad (5.18c)$$

$$+ \mathbf{P}_{112} - \mathbf{P}_{122}\mathbf{P}_{222} - \mathbf{P}_{222}\mathbf{P}_{122} - \mathbf{\Sigma}_{122} = \mathbf{0},$$

$$\mathbf{Q}_{211} - \mathbf{P}_{212}\mathbf{P}_{212}^\top - \frac{1}{\gamma^2}(\mathbf{P}_{112}\mathbf{P}_{212}^\top + \mathbf{P}_{212}\mathbf{P}_{112}^\top) + \mathbf{\Sigma}_{211} = \mathbf{0}, \quad (5.19a)$$

$$\mathbf{Q}_{212} - \mathbf{P}_{212}\mathbf{P}_{222} + \mathbf{P}_{211} - \frac{1}{\gamma^2}(\mathbf{P}_{112}\mathbf{P}_{222} + \mathbf{P}_{212}\mathbf{P}_{122}) + \mathbf{\Sigma}_{212} = \mathbf{0}, \quad (5.19b)$$

$$\mathbf{Q}_{222} - \mathbf{P}_{222}^2 + \mathbf{P}_{212}^\top + \mathbf{P}_{212} - \frac{1}{\gamma^2}(\mathbf{P}_{122}\mathbf{P}_{222} + \mathbf{P}_{222}\mathbf{P}_{122}) + \mathbf{\Sigma}_{222} = \mathbf{0}. \quad (5.19c)$$

Algebraic $\bar{P}$ matrix solutions, i.e. matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ satisfying (5.18) and (5.19), are not unique, and determining the block elements of $\mathbf{P}_i$ in (5.17) is non-trivial. In the following, we will show that choosing $\mathbf{P}_{112} = -k\mathbf{P}_{212}$, with $k > 0$, enables us to find an algebraic $\bar{P}$ solution for the game defined in Problem 5.2.

Inserting $\mathbf{P}_{112} = -k\mathbf{P}_{212}$ into (5.18a) and (5.19a) gives

$$\left(2k - 1 - \frac{k^2}{\gamma^2}\right)\mathbf{P}_{212}\mathbf{P}_{212}^\top = \mathbf{Q}_{111} + \mathbf{\Sigma}_{111} \quad (5.20a)$$

$$\left(1 - \frac{2k}{\gamma^2}\right)\mathbf{P}_{212}\mathbf{P}_{212}^\top = \mathbf{Q}_{211} + \mathbf{\Sigma}_{211}. \quad (5.20b)$$

Since $\mathbf{Q}_{i11} + \mathbf{\Sigma}_{i11} \geq 0$, in order for a solution of (5.20) to exist γ and k should satisfy

$$2k - 1 - \frac{k^2}{\gamma^2} > 0, \quad (5.21a)$$

$$1 - \frac{2k}{\gamma^2} > 0 \quad (5.21b)$$

which requires k to be chosen such that

$$\gamma^2 - \gamma\sqrt{\gamma^2 - 1} < k < \frac{\gamma^2}{2}. \quad (5.22)$$

Notice that in order for the interval in (5.22) from which k can be chosen to be non-empty, γ must satisfy

$$\gamma > \frac{2}{\sqrt{3}}. \quad (5.23)$$

Remark. When $\gamma > 1$, the inequality (5.4) merely limits how much the disturbance $\mathbf{w}$ is amplified in the performance variable z_1 , but does not necessarily mean that the disturbance is suppressed. A solution admitting lower values for γ may be found by introducing lower weights on the input $\mathbf{u}$ in the cost functionals (5.5), (5.6), to weigh it less compared to the state $\mathbf{x}$ and disturbance $\mathbf{w}$, as was done in [32]. A similar effect can be achieved by instead increasing the elements of the matrix $\mathbf{Q}_1$.

Let k then be chosen such that $2k - 1 - \frac{k^2}{\gamma^2} = 1 - \frac{2k}{\gamma^2}$. The solution k which also satisfies (5.22) is then

$$k = 1 + \gamma^2 - \sqrt{1 + \gamma^4}. \quad (5.24)$$

Now let $a = \frac{1}{1 - \frac{2k}{\gamma^2}}$. Choosing $\Sigma_{111} = \mathbf{Q}_{211} + b_1 \mathbf{I}_n$ and $\Sigma_{211} = \mathbf{Q}_{111} + b_1 \mathbf{I}_n$, with $b_1 > 0$, gives

$$\mathbf{P}_{212} \mathbf{P}_{212}^\top = a (\mathbf{Q}_{111} + \mathbf{Q}_{211} + b_1 \mathbf{I}_n). \quad (5.25)$$

Since the right-hand side of (5.25) is positive definite, $\mathbf{P}_{212}$ can be chosen as the Cholesky decomposition of the right-hand side, and will also be positive definite [14].

Remark. The closer γ is chosen to its lower bound (5.23), the greater a becomes. In order for the optimal control input (5.12) to be feasible, the inequality (5.23) should be satisfied with some margin.

Now let $\mathbf{P}_{122} = -c \mathbf{P}_{222}$ with $c > 0$. Equations (5.18c) and (5.19c) give

$$\left(2c - 1 - \frac{c^2}{\gamma^2}\right) \mathbf{P}_{222} = \Sigma_{122} + \mathbf{Q}_{122} + k (\mathbf{P}_{212}^\top + \mathbf{P}_{212}), \quad (5.26a)$$

$$\left(1 - \frac{2c}{\gamma^2}\right) \mathbf{P}_{222} = \mathbf{Q}_{222} + \Sigma_{222} + \mathbf{P}_{212}^\top + \mathbf{P}_{212}. \quad (5.26b)$$

Let $c = k$, and choose

$$\begin{aligned} \Sigma_{122} &= \mathbf{Q}_{222} + (1 - k) (\mathbf{P}_{212}^\top + \mathbf{P}_{212}) + b_2 \mathbf{I}_n \\ \Sigma_{222} &= \mathbf{Q}_{122} + b_2 \mathbf{I}_n \end{aligned} \quad (5.27)$$

with $b_2 > 0$. Since $k < 1$, Σ_{122} and Σ_{222} are both positive definite. We then have

$$\mathbf{P}_{222}^2 = a (\mathbf{Q}_{122} + \mathbf{Q}_{222} + b_2 \mathbf{I}_n + \mathbf{P}_{212}^\top + \mathbf{P}_{212}), \quad (5.28)$$

where the right-hand side is positive definite, and hence its root $\mathbf{P}_{222}$ exists and is likewise positive definite.

Finally, inserting $\mathbf{P}_{112} = -k \mathbf{P}_{212}$ and $\mathbf{P}_{122} = -k \mathbf{P}_{222}$ into equations (5.18b) and (5.19b) gives

$$\mathbf{P}_{111} = \Sigma_{112} + \mathbf{Q}_{112} - \frac{1}{a} \mathbf{P}_{212} \mathbf{P}_{222}, \quad (5.29a)$$

$$\mathbf{P}_{211} = -\Sigma_{212} - \mathbf{Q}_{212} + \frac{1}{a} \mathbf{P}_{212} \mathbf{P}_{222} \quad (5.29b)$$

where we choose $\Sigma_{112} = -\mathbf{Q}_{112}$, $\Sigma_{212} = -\mathbf{Q}_{212}$.

We have now found matrices $\mathbf{P}_1(\mathbf{x})$, $\mathbf{P}_2(\mathbf{x})$, $\Sigma_1(\mathbf{x})$, $\Sigma_2(\mathbf{x})$ which solve the equations (5.16) for a system with double integrator dynamics. To be an algebraic $\bar{P}$ matrix solution, $\mathbf{P}_1(\mathbf{x})$, $\mathbf{P}_2(\mathbf{x})$ must also satisfy $\mathbf{P}_2(\mathbf{0}) - \mathbf{P}_1(\mathbf{0}) > 0$, and the matrices

$\Sigma_i(\mathbf{x})$ must satisfy $\Sigma_i(\mathbf{x}) \geq 0$ and $\Sigma_i(\mathbf{0}) > 0$ for $i = 1, 2$. The matrices $\Sigma_1(\mathbf{x})$ and $\Sigma_2(\mathbf{x})$ both have positive definite upper-left blocks $\Sigma_{i11}(\mathbf{x})$. They are then positive definite if they fulfil

$$\Sigma_{i22}(\mathbf{x}) - \Sigma_{i12}(\mathbf{x})^\top \Sigma_{i11}(\mathbf{x})^{-1} \Sigma_{i12}(\mathbf{x}) > 0. \quad (5.30)$$

Inserting for $\Sigma_1(\mathbf{x})$, and using that $\mathbf{P}_{212}(\mathbf{x})$ and $\mathbf{Q}_{211}(\mathbf{x})$ are positive (semi) definite gives

$$\begin{aligned} & \mathbf{Q}_{222}(\mathbf{x}) + (1 - k) (\mathbf{P}_{212}(\mathbf{x})^\top + \mathbf{P}_{212}(\mathbf{x})) + b_2 \mathbf{I}_n \\ & - \mathbf{Q}_{112}(\mathbf{x})^\top (\mathbf{Q}_{211}(\mathbf{x}) + b_1 \mathbf{I}_n)^{-1} \mathbf{Q}_{112}(\mathbf{x}) \\ & \geq \mathbf{Q}_{222}(\mathbf{x}) + b_2 \mathbf{I}_n - \frac{1}{b_1} \mathbf{Q}_{112}(\mathbf{x})^\top \mathbf{Q}_{112}(\mathbf{x}). \end{aligned} \quad (5.31)$$

By Assumption 5.2 and since $\mathbf{Q}_{222}(\mathbf{x})$ is positive semidefinite, this is positive definite and so $\Sigma_1(\mathbf{x})$ is positive definite. Likewise, $\Sigma_2(\mathbf{x})$ can also be shown to be positive definite.

Consider now $\mathbf{P}_2(\mathbf{x})$, with $\mathbf{P}_{211}(\mathbf{x})$ given by (5.29b). Since $\frac{1}{a} > 0$ and both $\mathbf{P}_{212}(\mathbf{x})$ and $\mathbf{P}_{222}(\mathbf{x})$ are positive definite, so is their product $\mathbf{P}_{211}(\mathbf{x})$ [14]. Then $\mathbf{P}_2(\mathbf{x}) > 0$ if the following holds:

$$\mathbf{P}_{222}(\mathbf{x}) - \mathbf{P}_{212}(\mathbf{x})^\top \mathbf{P}_{211}(\mathbf{x})^{-1} \mathbf{P}_{212}(\mathbf{x}) > 0 \quad (5.32)$$

Substituting (5.29b) into (5.32) gives

$$(\mathbf{P}_{222}(\mathbf{x})^2 - a \mathbf{P}_{212}(\mathbf{x})^\top) \mathbf{P}_{222}(\mathbf{x})^{-1} > 0. \quad (5.33)$$

Since $\mathbf{P}_{222}(\mathbf{x})$ is positive definite, so is its inverse. By rearranging (5.28) to find an expression for $\mathbf{P}_{222}(\mathbf{x})^2 - a \mathbf{P}_{212}(\mathbf{x})^\top$ we see that this term is also positive definite. Hence (5.33) holds, and $\mathbf{P}_2(\mathbf{x})$ is positive definite for all $\mathbf{x}$.

The matrix $\mathbf{P}_1(\mathbf{x})$ is negative definite if $-\mathbf{P}_1(\mathbf{x})$ is positive definite. Since $-\mathbf{P}_{111}(\mathbf{x}) > 0$, $-\mathbf{P}_1(\mathbf{x})$ is positive definite if

$$\begin{aligned} & -\mathbf{P}_{122}(\mathbf{x}) + \mathbf{P}_{112}(\mathbf{x})^\top \mathbf{P}_{111}^{-1}(\mathbf{x}) \mathbf{P}_{112}(\mathbf{x}) = \\ & k \mathbf{P}_{222}(\mathbf{x}) - k^2 \mathbf{P}_{212}(\mathbf{x})^\top \mathbf{P}_{211}(\mathbf{x})^{-1} \mathbf{P}_{212}(\mathbf{x}) > 0. \end{aligned} \quad (5.34)$$

Again inserting (5.29b) for $\mathbf{P}_{211}(\mathbf{x})$, this can be rewritten as

$$(\mathbf{P}_{222}(\mathbf{x})^2 - ak \mathbf{P}_{212}(\mathbf{x})^\top) \mathbf{P}_{222}(\mathbf{x})^{-1} > 0 \quad (5.35)$$

Since $k < 1$, $\mathbf{P}_{212}(\mathbf{x})$ is positive definite and the inequality (5.33) holds, the inequality (5.35) also holds. Since $\mathbf{P}_2(\mathbf{x})$ and $-\mathbf{P}_1(\mathbf{x})$ are positive definite for all $\mathbf{x}$, it follows that $\mathbf{P}_2(\mathbf{0}) - \mathbf{P}_1(\mathbf{0}) > 0$.

The result of this section is summarized in the following

Lemma 5.1. *Consider the system (5.13) with output variables (5.2), where $\mathbf{Q}_1(\mathbf{x})$, $\mathbf{Q}_2(\mathbf{x})$ are subject to Assumption 5.2 and with $\gamma > \frac{2}{\sqrt{3}}$. The matrices $\mathbf{P}_1(\mathbf{x})$, $\mathbf{P}_2(\mathbf{x})$ with $\mathbf{P}_2$ given by (5.25), (5.28), (5.29b), and $\mathbf{P}_1(\mathbf{x})$ given by (5.29a) and*

$$\mathbf{P}_{112}(\mathbf{x}) = -k \mathbf{P}_{212}(\mathbf{x}), \quad \mathbf{P}_{122}(\mathbf{x}) = -k \mathbf{P}_{222}(\mathbf{x}) \quad (5.36)$$

constitute an algebraic $\bar{P}$ solution for Problem 5.2.

5.4 Application to AIAUVs

The result from Section 5.3 will be combined with a feedback linearizing controller in order to apply it to AIAUVs with dynamics like described in Section 2.1.1.

5.4.1 Feedback linearization of joint dynamics

In this chapter we consider only the joint motion of the AIAUV. The position and orientation of the base are assumed to be controlled using the thrusters independently of the joint motion, using control methods such as those in e.g. [18].

The dynamics of the joints alone are described by the last n rows of the dynamics (2.1b), which can be written as

$$\ddot{\theta} = [\mathbf{0}_{n \times 6} \mathbf{I}_n] \mathbf{M}(\theta)^{-1} \left(\begin{bmatrix} \mathbf{B}_{\text{thr}}(\theta) \\ \mathbf{B}_\theta(\theta) \end{bmatrix} \tau_{\text{thr}} + \begin{bmatrix} \mathbf{0}_{6 \times n} \\ \mathbf{I}_n \end{bmatrix} \tau_\theta - \mathbf{C}(\theta, \zeta) \dot{\zeta} - \mathbf{D}(\theta, \zeta) \dot{\zeta} - \mathbf{g}(\mathbf{q}, \theta) \right) \quad (5.37)$$

Assume now that only an estimate of the parameters of the manipulator inertia matrix is available. The resulting estimated inertia matrix $\hat{\mathbf{M}}(\theta)$ must be positive definite. One such choice of estimate could be the rigid body mass only, as it is often more easily available, while hydrodynamic parameters due to added mass effects are difficult to identify.

Let $\tilde{\mathbf{M}}(\theta) = \mathbf{M}(\theta) - \hat{\mathbf{M}}(\theta)$ be the error between the inertia matrix and its estimate. Further, let the lower-right $n \times n$ -block of $\hat{\mathbf{M}}^{-1}(\theta)$ be denoted $\hat{\mathbf{M}}_{\text{inv},\theta}(\theta)$, such that (5.37) can be written as

$$\ddot{\theta} = \hat{\mathbf{M}}_{\text{inv},\theta}(\theta) \tau_\theta + [\mathbf{0}_{n \times 6} \mathbf{I}_n] \hat{\mathbf{M}}(\theta)^{-1} \left(\begin{bmatrix} \mathbf{B}_{\text{thr}}(\theta) \\ \mathbf{B}_\theta(\theta) \end{bmatrix} \tau_{\text{thr}} - \mathbf{C}(\theta, \zeta) \dot{\zeta} - \mathbf{D}(\theta, \zeta) \dot{\zeta} - \mathbf{g}(\mathbf{q}, \theta) - \tilde{\mathbf{M}}(\theta) \dot{\zeta} \right) \quad (5.38)$$

Since the mass matrix estimate $\hat{\mathbf{M}}(\theta)$ is positive definite, the inverse and its lower-right block $\hat{\mathbf{M}}_{\text{inv},\theta}(\theta)$ are likewise positive definite, and hence invertible.

Assume further that the parameters of the thrust configuration matrix $\mathbf{B}(\theta)$ and hydrostatic forces $\mathbf{g}(\mathbf{q}, \theta)$ are known exactly, whereas for the parameters of $\mathbf{C}(\theta, \zeta)$ and $\mathbf{D}(\theta, \zeta)$ only estimates are available, due to the presence of uncertain hydrodynamic effects. In particular, the estimated matrix $\hat{\mathbf{C}}(\theta, \zeta)$ should be computed from the estimate $\hat{\mathbf{M}}(\theta)$. Denote the estimated damping matrix $\hat{\mathbf{D}}(\theta, \zeta)$, let $\mathbf{u}$ now denote a new, virtual input to be assigned later, and let

$$\tau_\theta = \hat{\mathbf{M}}_{\text{inv},\theta}(\theta)^{-1} \left(\mathbf{u} - [\mathbf{0}_{n \times 6} \mathbf{I}_n] \hat{\mathbf{M}}(\theta)^{-1} \left(\mathbf{B}(\theta) \tau_{\text{thr}} + \hat{\mathbf{C}}(\theta, \zeta) \dot{\zeta} + \hat{\mathbf{D}}(\theta, \zeta) \dot{\zeta} - \mathbf{g}(\mathbf{q}, \theta) \right) \right) \quad (5.39)$$

Remark. The parameters of the thrust configuration matrix and hydrostatic forces should be known exactly in order for the disturbance due to parameter uncertainties to have bounded energy, as considered in Problem 5.1. In some cases also the parameters of $\mathbf{g}(\mathbf{q}, \boldsymbol{\theta})$ may be uncertain, e.g. in the presence of tunnel thrusters through links, making it difficult to know their volume exactly. The simulation results in [32], where an error in the gravitational term is included, suggests some robustness to this type of errors as well.

Inserting the input (5.39) into the dynamics (5.38), and introducing error terms $\tilde{\mathbf{C}}(\boldsymbol{\theta}, \boldsymbol{\zeta}) = \mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\zeta}) - \hat{\mathbf{C}}(\boldsymbol{\theta}, \boldsymbol{\zeta})$ and $\tilde{\mathbf{D}}(\boldsymbol{\theta}, \boldsymbol{\zeta}) = \mathbf{D}(\boldsymbol{\theta}, \boldsymbol{\zeta}) - \hat{\mathbf{D}}(\boldsymbol{\theta}, \boldsymbol{\zeta})$, yields

$$\ddot{\boldsymbol{\theta}} = \mathbf{u} + \underbrace{[\mathbf{0}_{n \times 6} \quad \mathbf{I}_n] \hat{\mathbf{M}}(\boldsymbol{\theta})^{-1} \left(-\tilde{\mathbf{C}}(\boldsymbol{\theta}, \boldsymbol{\zeta})\boldsymbol{\zeta} - \tilde{\mathbf{D}}(\boldsymbol{\theta}, \boldsymbol{\zeta})\boldsymbol{\zeta} - \tilde{\mathbf{M}}(\boldsymbol{\theta})\dot{\boldsymbol{\zeta}} \right)}_{=\mathbf{w}} \quad (5.40)$$

The dynamics of the joint angles $\boldsymbol{\theta}$ can then be written as

$$\begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_n \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_n \end{bmatrix} \mathbf{u} \quad (5.41)$$

Finally, in order to achieve setpoint regulation, let $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \boldsymbol{\theta}_d$, where $\boldsymbol{\theta}_d$ is a vector of constant, desired joint setpoints. Then

$$\dot{\tilde{\boldsymbol{\theta}}} = \dot{\boldsymbol{\theta}}, \quad \ddot{\tilde{\boldsymbol{\theta}}} = \ddot{\boldsymbol{\theta}} \quad (5.42)$$

which gives

$$\begin{bmatrix} \dot{\tilde{\boldsymbol{\theta}}} \\ \ddot{\tilde{\boldsymbol{\theta}}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\theta}} \\ \dot{\tilde{\boldsymbol{\theta}}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_n \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_n \end{bmatrix} \mathbf{u} \quad (5.43)$$

since the linearized system dynamics (5.41) do not depend on the original state $\boldsymbol{\theta}$. The control input (5.39) still requires the original state $\boldsymbol{\theta}$, which can be found as $\boldsymbol{\theta} = \tilde{\boldsymbol{\theta}} + \boldsymbol{\theta}_d$. We will now let $\mathbf{x} = \begin{bmatrix} \tilde{\boldsymbol{\theta}}^\top & \dot{\tilde{\boldsymbol{\theta}}}^\top \end{bmatrix}^\top$ denote the error state of the system, and note that it satisfies double integrator dynamics (5.13).

The virtual input $\mathbf{u}$ will be selected as the optimal control input in (5.12).

5.4.2 Performance variables

We wish to design the cost functional (5.5) to be minimized by the virtual control input $\mathbf{u}$ such that it leads to a trajectory along which collisions between links of the AIAUV are avoided, similarly to what has been done in [108] for collision avoidance between multiple agents. In addition, joint limits should be obeyed.

The matrix $\mathbf{Q}_1(\mathbf{x})$ will be chosen simply as a constant, diagonal matrix. A natural choice is the identity matrix $\mathbf{I}_{2n}$.

For the matrix $\mathbf{Q}_2(\mathbf{x})$, the lower right block $\mathbf{Q}_{222}(\mathbf{x})$ will be chosen simply as a diagonal matrix. The upper-left block $\mathbf{Q}_{211}(\mathbf{x})$ will be a diagonal matrix with elements $\mathbf{Q}_{211_i}(\mathbf{x})$, $i = 1, \dots, n$ chosen as

$$\mathbf{Q}_{211_i}(\mathbf{x}) = \alpha_i + \beta_i^l g_i^l(\theta_i) + \beta_i^c g_i^c(\boldsymbol{\theta}) \quad (5.44)$$

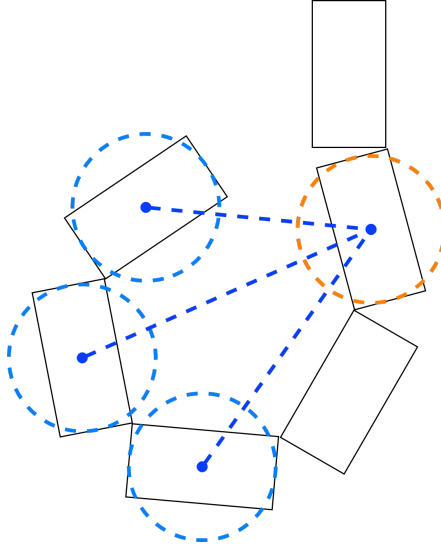


Figure 5.1: Illustration of distances to be considered for avoiding collisions with a link i (orange)

where α_i , β_i^l , β_i^c are positive weights, $g_i^l(\theta_i)$ is a joint limit avoidance function for joint i , and $g^c(\boldsymbol{\theta})$ is a collision avoidance function penalizing collisions between links of the AIAUV.

The joint limit avoidance function $g_i^l(\theta_i)$ is chosen as

$$g_i^l(\theta_i) = \frac{1}{(\theta_i^2 - \theta_{\text{lim},i}^2)^2} \quad (5.45)$$

where $\theta_{\text{lim},i}$ is the absolute value of the maximum joint angle allowed for joint i , typically due to physical limitations.

Collision avoidance between links will be achieved similarly as in the case of separate agents in [108], by using inverse barrier functions as weights in the matrix $\mathbf{Q}_2(\mathbf{x})$. Each joint may cause collisions between not only the links which it connects, but any two links on opposite sides of the joint. Therefore, creating separate functions including only the collisions a given joint may cause would give a large overlap between the individual functions. To simplify this, all joint errors will be weighted by the same function which takes into account all possible collisions.

Denote by l_i half of the length of link i . A sphere of radius l_i can then almost encompass the link in the case of cylindrical links. To simplify the collision avoidance function, this sphere will be used as an approximation of the shape of the link. A collision between two links i and j can then be avoided by ensuring that the distance between their centers is greater than $l_i + l_j$. A less conservative behaviour can be achieved by approximating the link shapes as ellipsoids instead. In the case of neighbouring links, collisions can easily be avoided simply by respecting appropriate joint angle limits, and will therefore not be included in the collision avoidance function. This is illustrated in Figure 5.1.

The distance between the centers of two links can be found using homogeneous transformations. As described in Section 2.1, each link has an associated reference frame attached to it. The homogeneous transformation from frame i to $i + 1$, for $i = 1, \dots, n$ is

$$\mathbf{T}_{i+1}^i = \begin{bmatrix} & 2l_i \\ \mathbf{R}_{\theta_i} & 0 \\ & 0 \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \quad (5.46)$$

where $\mathbf{R}_{\theta_i}$ is the rotation matrix describing the rotation caused by joint i . Consequently, the position p_j^i of the center of link j relative to the center of link i in homogeneous coordinates, with $j > i$, can be found as

$$\mathbf{p}_j^i = \begin{bmatrix} & l_i \\ \mathbf{R}_{\theta_i} & 0 \\ & 0 \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \mathbf{T}_{i+2}^{i+1} \dots \mathbf{T}_j^{j-1} \begin{bmatrix} l_j \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (5.47)$$

Then the distance between the links i and j , squared, is $d_{ij}(\boldsymbol{\theta}) = \mathbf{p}_j^{i\top} \mathbf{p}_j^i - 1$. We now define the collision avoidance function as

$$g^c(\boldsymbol{\theta}) = \sum_{i=1}^{n+1} \sum_{j=i+2}^{n+1} \frac{1}{(d_{ij}(\boldsymbol{\theta}) - (l_i + l_j)^2)^2}, \quad (5.48)$$

where it has been assumed that collision between adjacent links can be avoided by setting appropriate joint limits.

As in [108], neither the initial state $\boldsymbol{\theta}(0)$ nor the desired state $\boldsymbol{\theta}_d$ can be states in which a collision occurs, and there must exist a collision-free path from $\boldsymbol{\theta}(0)$ to $\boldsymbol{\theta}_d$ in the state-space. Collision avoidance with obstacles in the environment can be similarly achieved by taking into account the position of the base relative to the obstacle.

The matrices $\mathbf{Q}_1(\mathbf{x})$, $\mathbf{Q}_2(\mathbf{x})$ have now been chosen such that $\mathbf{Q}_1(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})$ is positive definite and hence has full rank. Then if the output $y = \mathbf{x}^\top \mathbf{Q}_1(\mathbf{x}) \mathbf{x} + \mathbf{x}^\top \mathbf{Q}_2(\mathbf{x}) \mathbf{x}$ is identically equal to zero, so is the state $\mathbf{x}$, and Assumption 5.1 is satisfied for the system for any choice of $\mathbf{Q}_1(\mathbf{x})$, $\mathbf{Q}_2(\mathbf{x})$ such that their sum is positive definite. Then using the algebraic $\bar{P}$ matrix solutions for the double integrator dynamics from Section 5.3.1, Theorem 5.1 states that the system (5.43) is asymptotically stable if $\mathbf{u}$, $\mathbf{w}$ are the optimal input and worst-case disturbance, respectively, as given by (5.12) with dynamic extension $\boldsymbol{\xi}$ as given by (5.11) and V_1 , V_2 as in (5.9).

5.5 Simulation results

The control scheme described in Section 5.4 is implemented on the vehicles described in Section 2.1.2. The even-numbered links are short links separating two joints, simulating a physical vehicle with 2DOF-joints and 5 actual links. The collision function (5.48) is therefore implemented such that it only considers the odd-numbered links.

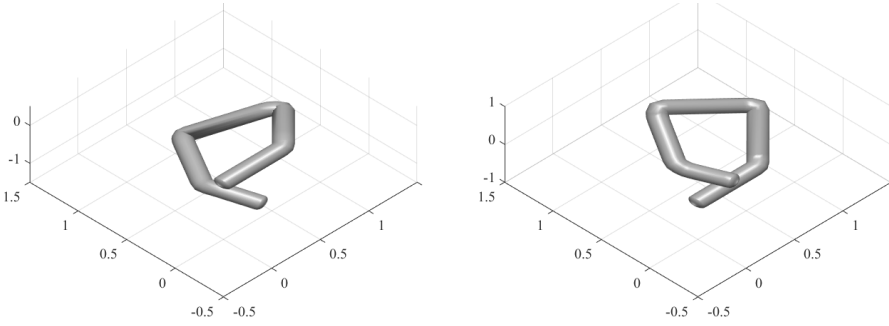


Figure 5.2: Initial (left) and desired final (right) joint configuration.

The initial joint configuration $\boldsymbol{\theta}(0)$ and desired setpoint is $\boldsymbol{\theta}_d$ are chosen as

$$\begin{aligned}\boldsymbol{\theta}(0) &= \left[\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{6} \right]^\top, \\ \boldsymbol{\theta}_d &= \left[\frac{\pi}{4}, -\frac{\pi}{6}, \frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{4}, -\frac{\pi}{6} \right]^\top,\end{aligned}$$

and all initial joint velocities are 0. The initial and desired configurations are shown in Figure 5.2. Motions like this may for instance arise if the vehicle has attached one end to a structure, while the other end has to interact with multiple points on the structure.

The initial value $\boldsymbol{\xi}(0)$ is chosen by selecting a start value $\boldsymbol{\xi}_0$, then finding the nearest value for $\boldsymbol{\xi}(0)$ such that the inequalities (5.10) are satisfied at $t = 0$, and $\boldsymbol{\xi}(0) + \boldsymbol{\theta}_d$ satisfy joint limits and there are no collisions between links. The search is performed using the MATLAB function `fmincon` to minimize the norm of the error $\boldsymbol{\xi}(0) - \boldsymbol{\xi}_0$ with $\boldsymbol{\xi}(0)$ subject to the aforementioned constraints. The search was initiated at

$$\boldsymbol{\xi}_0 = \left[-\frac{\pi}{2}, \frac{\pi}{3}, -\pi, \frac{\pi}{3}, -\pi, \frac{\pi}{3}, -\frac{\pi}{2}, \frac{\pi}{3}, -10, 0, -10, 0, -10, 0, -10, 0 \right]^\top,$$

and the resulting final choice of $\boldsymbol{\xi}(0)$ was such that $\|\boldsymbol{\xi}(0) - \boldsymbol{\xi}_0\| = 0.07$.

The physical joint limits of all joints are $\pm \frac{\pi}{2}$, and the limits in (5.45) are set to $\theta_{\text{lim},i} = \frac{\pi}{2} + 0.1$ to allow the full range of the joints to be used. The parameters of the cost functionals (5.6), (5.5) are chosen as $\gamma = 1.2$, $\mathbf{Q}_1(\mathbf{x}) = \mathbf{I}_{2n}$, $\mathbf{Q}_{222}(\mathbf{x}) = \frac{1}{2}\mathbf{I}_n$, and $\mathbf{Q}_{211}(\mathbf{x})$ with elements as given by (5.44) with $\alpha_i = \frac{1}{2}$, $\beta_i^l = 0.1$ and $\beta_i^c = 0.01$ for all i . The parameters of the algebraic $\bar{P}$ solution are chosen to be $b_1 = 4.5$, $b_2 = 1.5$, and the parameters of $\dot{\boldsymbol{\xi}}$ and the value functions (5.9) are selected as $\kappa = 1$, $\mathbf{R}_1 = \begin{bmatrix} 1.1 & -1 \\ -1 & 1.1 \end{bmatrix}$ and $\mathbf{R}_2 = \begin{bmatrix} 1.2 & 1 \\ 1 & 1.2 \end{bmatrix}$. No thrust is used in the simulations, causing the AIAUV to float freely and rotate due to the joint motion.

Simulations are performed for three cases:

- 1) With $\mathbf{w} = \mathbf{w}^*$,
- 2) With $\hat{\mathbf{M}}(\boldsymbol{\theta}) = 0.5\mathbf{M}(\boldsymbol{\theta})$, $\hat{\mathbf{C}}(\boldsymbol{\theta}, \boldsymbol{\zeta}) = 0.5\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\zeta})$ and $\hat{\mathbf{D}}(\boldsymbol{\theta}, \boldsymbol{\zeta}) = 0.5\mathbf{D}(\boldsymbol{\theta}, \boldsymbol{\zeta})$ in the controller (5.39),
- 3) With estimates as above, but with $\mathbf{Q}_1(\mathbf{x}) = 10\mathbf{I}_{2n}$.

The simulation results are shown in Figures 5.3, 5.4, 5.5 for cases 1, 2 and 3, respectively. In Figure 5.3, the behaviour of joints 1 and 7, 3 and 5, and 2, 4, 6 and

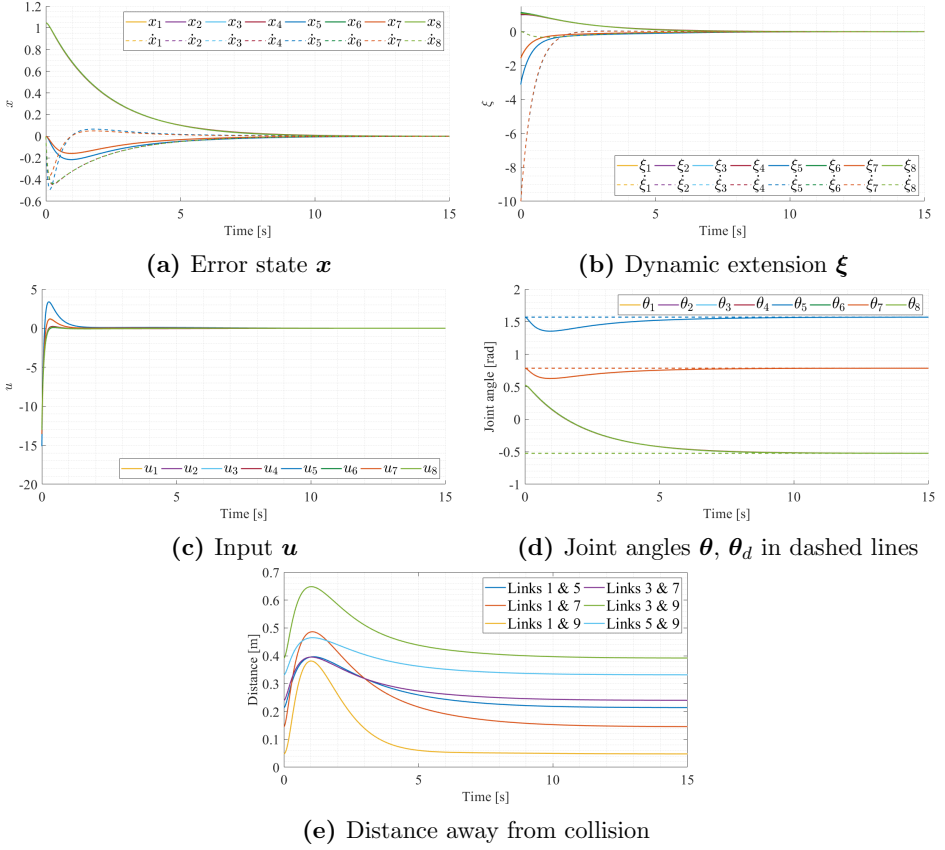


Figure 5.3: Simulation results with $w = w^*$

8 coincides. The inequalities (5.10) are fulfilled in the first simulation case, but not cases 2 and 3. In case 3, only $\mathcal{HJI}_1$ (5.10a) does not fulfil the inequality.

The joint angle errors converge to zero while successfully avoiding collision between links in all the simulated cases, as can be seen from Figures 5.3, 5.4 and 5.5, despite Theorem 5.1 only stating that the system is asymptotically stable when $w = w^*$. While γ is the same in all cases, comparing Figures 5.4a and 5.5a show that better disturbance attenuation and faster convergence can be achieved by increasing $Q_1(x)$, which corresponds to reducing both γ and the weight on the input u in the cost functional J_1 (5.6). This way, disturbance attenuation can be improved despite there being a lower bound on γ , but at the expense of greater control effort, as can be seen in Figures 5.4b and 5.5b.

The inequalities (5.10) are not always satisfied in the cases when the disturbance w is not the worst-case w^* . When the inequalities are not satisfied, the $\mathcal{L}_2$ -stability with respect to the disturbance w is no longer guaranteed, and neither is boundedness of $h_2(x)^\top h_2(x)$, which is what ensures collision avoidance. While Theorem 5.1 states that there exists a region Ω in which the inequalities (5.10) are satisfied, it gives no guarantees that the solution will remain in Ω . Whether the system trajec-

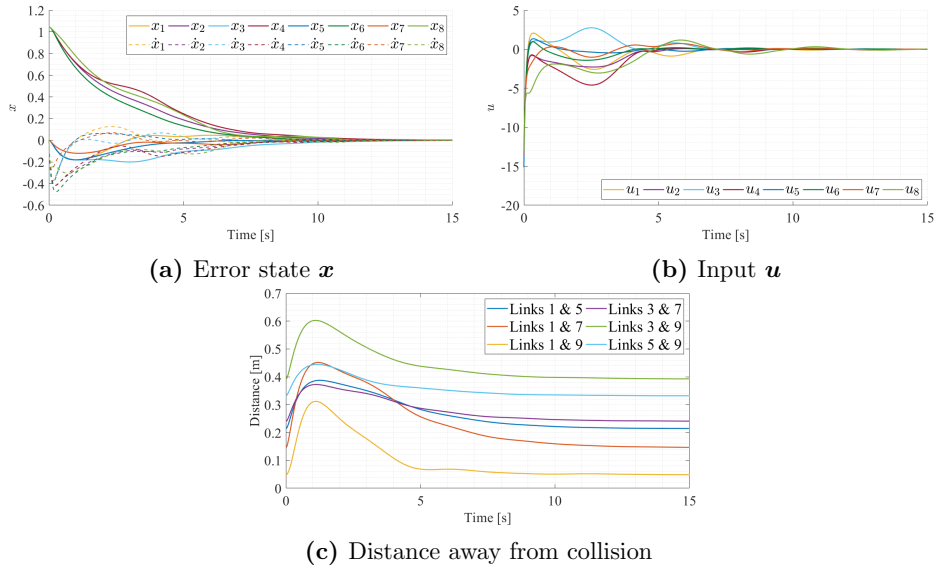


Figure 5.4: Simulation results with imperfect estimates of parameters

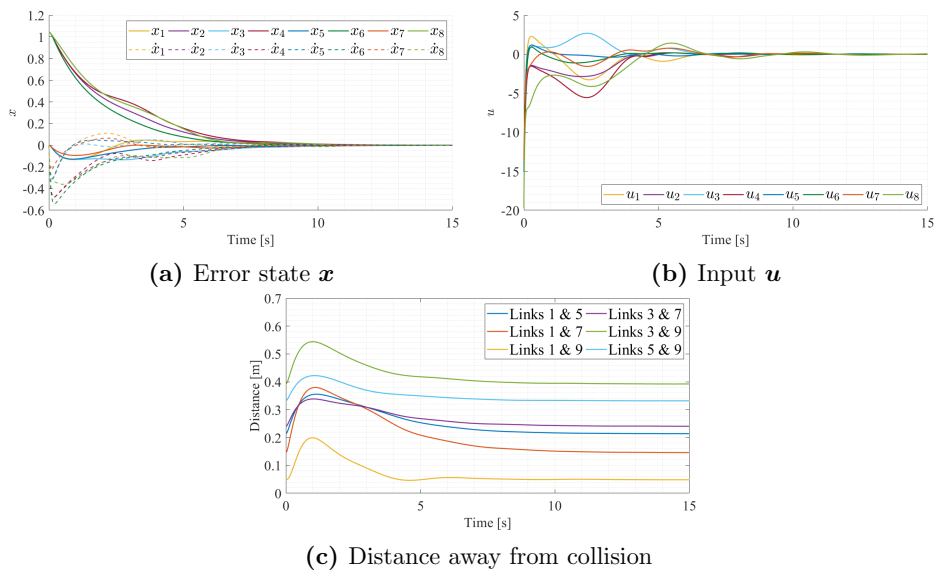


Figure 5.5: Simulation results with imperfect estimates of parameters and $Q_1(x) = 10I_{2n}$

ories remain in Ω depends, at least in part, on the choice of $\xi(0)$. Finding such a $\xi(0)$ is not easy and often relies on much trial and error. In addition, a new $\xi(0)$ is required for every new motion to be performed. This difficulty can be overcome by following an approach similar to that presented in [110]. Namely, global asymptotic stability can be achieved by enabling ξ to evolve according to hybrid dynamics, enabling ξ to jump to a new value such that the inequalities (5.10) remain satisfied. This may come at the cost of discontinuities in the input, which is difficult to realize and causes strain on physical actuators.

The task considered here is setpoint regulation of joints. If the true goal is to e.g. control the end-effector position, there may be multiple joint configurations which give the same end-effector position. In the case of obstacles in the environment, some of these configurations might be impossible to reach from a given initial configuration, as pointed out in [11]. The selection of desired setpoints therefore requires careful planning.

Applying the control torque (5.39) to the original dynamics with $\mathbf{u} = \mathbf{u}^*$ as given by (5.12) will result in a disturbance $\mathbf{w}$ which is not necessarily the worst-case disturbance $\mathbf{w}^*$ given by (5.12). Since the strategies (5.12) are the solution to a modified differential game, they have the property that $\tilde{J}_1(\mathbf{u}^*, \mathbf{w}) \geq \tilde{J}_1(\mathbf{u}^*, \mathbf{w}^*)$, with $\tilde{J}_1(\mathbf{u}, \mathbf{w})$ being the modified version of the cost functional (5.6), which is such that $\tilde{J}_1(\mathbf{u}, \mathbf{w}) \leq J_1(\mathbf{u}, \mathbf{w})$ [107]. Hence the robustness properties of the full joint dynamics (5.38) with input (5.39) and virtual input $\mathbf{u} = \mathbf{u}^*$ will depend on the margin in the first of the inequalities in (5.10).

The cost function (5.5) depends only on the system state $\mathbf{x}$. In order to incorporate collision avoidance with dynamic obstacles, the cost function would either have to become time-varying, or the obstacles would need to be considered part of the state. The latter would mean that in designing the control input it is implicitly assumed that the obstacles also have double integrator dynamics, that they collaborate on minimizing the cost function (5.5), and that they do so by also applying the optimal control input. Similarly as with the disturbance, in a real-life scenario the encountered dynamic obstacles would in fact not behave optimally, breaking with the premises on which the optimal input was derived. In consequence, it would not be possible to guarantee that collision will be avoided in practice. An alternative approach may be to include dynamic obstacles as additional players with their own cost functions, as in [108, 109]. These cost functions could then be designed to describe a worst-case scenario. The resulting control input would still be derived on the premise that both obstacles and disturbance play optimally, but the optimal scenario may now be closer to a real-life one. However, in [108, 109] the cost functional of each agent includes only the respective agent's control input. Combining multiple agents with a disturbance, which acts as an actively hostile player in that it aims to maximize another players control effort, will require solving a more complicated set of equations than considered either in this chapter or in [108, 109].

5.6 Chapter summary

In this chapter, an algebraic $\bar{P}$ matrix solution was found for a system satisfying double integrator dynamics and nonlinear cost functionals in order to construct an

approximate solution to a mixed H_2/H_∞ control problem, formulated as a differential game. Then, using feedback linearization, the joint dynamics of an AIAUV were simplified into those of a double integrator, and the solution for mixed H_2/H_∞ control was applied to the task of joint setpoint regulation while avoiding joint limits and collisions between the links of the AIAUV. Simulation results show that collisions are successfully avoided while regulating the joints to desired setpoints, also in the case of parameter uncertainties in the feedback linearizing controller. The solution found in Section 5.3 is applicable to any system which can be feedback linearized into double integrator dynamics.

Chapter 6

Path Following using Geometric Control of Reduced Attitude

In this chapter, a control law for tracking of a reduced attitude is proposed for a general system with only two control torques, and the control law is shown to make the tracking error go to zero exponentially. The control law is used as a basis for controlling the pointing direction of an AIAUV by means of its joints. Combined with a LOS guidance law, this results in a path-following method for AIAUVs moving in 3D. The method is verified through simulations.

Contributions of this chapter

The first contribution of this chapter is the development and analysis of a tracking control law for reduced attitude for rigid bodies controlled by only two control torques. The control law is inspired by [23], but takes into account system inertia and control by means of torques instead of angular accelerations as in [23]. The proposed control law is proved to make the tracking error dynamic uniformly stable at the origin, and to make the tracking errors converge exponentially to zero provided that the body velocity about the unactuated body axis is bounded.

Second, a path following controller for AIAUVs proposed for the case of straight paths, and its performance validated in simulations. Unlike previous path following control methods for AIAUVs which consider only planar motion [16, 135, 149], the control method proposed in this chapter considers motion in 3D.

Publications

This chapter is based on [157].

Organization of this chapter

This chapter is structured as follows: a brief introduction is given in Section 6.1, and in Section 6.2, the reduced attitude tracking control for a general system is presented and analysed. Section 6.3 presents the path-following and pointing direction control methods for AIAUVs, and simulation results are shown in Section 6.4. Finally, a summary is given in Section 6.5.

6.1 Introduction

In this chapter, the travel direction of the AIAUV is controlled by using its joints to curve its body, as in [135] for an AIAUV equipped with tail thrusters only, or as for USRs in e.g. [76, 86, 92]. The considered AIAUVs have joints actuated in two directions, such that they can be used to point the front of the AIAUV in a desired direction. This approach has been chosen with the aim of consuming less energy than previous path-following and trajectory tracking methods for AIAUVs like those of [16, 149], by avoiding the additional use of thrusters for turning.

The desired travel direction is determined based on a guidance principle designed to guide a vehicle towards and onto a path to be followed. A guidance principle frequently used in path following for AUVs is the LOS principle [49], which aims the vehicle towards a point further ahead along the path. In 3D, this provides only a pointing direction, not a full attitude reference. Many AUVs are passively stabilized in roll, in which case the control problem can be simplified from a total of 6 DOFs to 5 by disregarding it as in [24], or a reference for the full attitude can be found [156]. However, due to the AIAUV's dual use as both a vehicle and a free-floating manipulator, it may not be desirable to heavily stabilize it, because this could impede its dexterity as a manipulator. Therefore it is beneficial if control of the pointing direction, or reduced attitude, of the AIAUV does not assume anything about roll.

The pointing direction of a vehicle or rigid body can be represented in terms of two Euler angles, most commonly pitch and yaw, or a unit vector directly. The Euler angle representation suffers from a singularity when the pitch angle is $\pm 90^\circ$, i.e. when the vehicle is travelling vertically upwards or downwards. The exact attitude at which the singularity occurs can be changed by making a different choice of rotations to represent the attitude, however, it will always be fixed relative to the world frame. In contrast, the unit vector representation is defined and differentiable everywhere on the two-sphere, i.e. at any attitude. While it is impossible to globally stabilize attitude, or reduced attitude, using a continuous controller [15], controllers based on vector representations (or matrix representations for full attitude) are typically able to make the attitude converge to its desired state from almost any initial condition, except the exact opposite of the desired attitude [31]. Therefore the point at which the controller fails is not fixed in the world frame.

Reduced attitude stabilization using a vector representation has been addressed by e.g. [31], while in [124, 125], the problem of reduced attitude tracking is addressed for spacecraft which are fully actuated in attitude. The third control input allows for additional tasks such as rotating about the pointing direction to be performed. In [101], the full attitude is stabilized by decomposing it into a reduced attitude and rotation about it, where stabilizing the reduced attitude part takes priority. More recently, control of reduced attitude has also been considered for fixed-wing unmanned aerial vehicles (UAVs) in [35], where the full attitude of the aircraft is controlled by decomposing into a reduced attitude, chosen to be direction of gravity given in the vehicle body frame, and rotation about it. Reduced attitude stabilization using only two control torques has been done by [23], as well as tracking using accelerations along the surface of the sphere as inputs. Geometric attitude control using a matrix representation of attitude has also been applied to marine vehicles,

to an underactuated surface vessel in [119], and to a spherical, fully actuated AUV in [134].

6.2 Reduced attitude tracking

In this section, a control law solving the problem of tracking a reduced attitude, or pointing direction, for a general system with only two control torques is proposed, and the control law is proven to make the tracking errors go to zero exponentially.

6.2.1 Preliminaries: Algebraic identities

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be vectors in $\mathbb{R}^3$. For the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ it holds that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}^\top \mathbf{c})\mathbf{b} - (\mathbf{a}^\top \mathbf{b})\mathbf{c}, \quad (6.1)$$

and

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c}). \quad (6.2)$$

The scalar triple product $\mathbf{a}^\top (\mathbf{b} \times \mathbf{c})$ is invariant under a circular shift:

$$\mathbf{a}^\top (\mathbf{b} \times \mathbf{c}) = \mathbf{b}^\top (\mathbf{c} \times \mathbf{a}) = \mathbf{c}^\top (\mathbf{a} \times \mathbf{b}). \quad (6.3)$$

6.2.2 Problem dynamics and kinematics

The attitude dynamics of a rigid body equipped with two control torques can be written as

$$\mathbf{J}\dot{\boldsymbol{\omega}}^b = \mathbf{m}(t, \boldsymbol{\omega}^b) + \mathbf{H}\mathbf{u} \quad (6.4)$$

where $\boldsymbol{\omega}^b \in \mathbb{R}^3$ is the angular velocity of the rigid body given in its body-fixed frame, $\mathbf{J} = \mathbf{J}^\top > 0$ is the 3×3 inertia matrix, $\mathbf{u} \in \mathbb{R}^2$ is the control input, and $\mathbf{m}(t, \boldsymbol{\omega}^b) \in \mathbb{R}^3$ represents all other moments acting on the system, which are assumed to be known. The matrix $\mathbf{H} = [\mathbf{e}_2 \ \mathbf{e}_3] \in \mathbb{R}^{3 \times 2}$ maps the two control torques to the actuated axes $\mathbf{e}_2$, $\mathbf{e}_3$ of the body-fixed frame.

Assumption 6.1. The inertia matrix $\mathbf{J}$ is assumed to have the block-diagonal structure

$$\mathbf{J} = \begin{bmatrix} J_{11} & 0 \\ 0 & \mathbf{J}_{22} \end{bmatrix} \quad (6.5)$$

where $J_{11} \in \mathbb{R}$, and $\mathbf{J}_{22} \in \mathbb{R}^{2 \times 2}$. Since $\mathbf{J}$ is a positive definite matrix, $\mathbf{J}_{22}$ is also positive definite [14].

Remark. Assumption 6.1 can be fulfilled either by aligning the body-frame axes with the principal axes of inertia of the rigid body, or by ensuring reflection symmetry through either the yz -plane, or both the xy - and xz -plane of the body frame [49].

The control objective is to align the unactuated axis of the body-fixed frame of the system with a reference pointing direction $\mathbf{x}_{\text{ref}}^I(t)$ given in an inertial frame I .

The reference pointing direction $\mathbf{x}_{\text{ref}}^I(t)$ has to satisfy $\|\mathbf{x}_{\text{ref}}^I(t)\| = 1 \forall t$. Its angular velocity expressed in the body frame is

$$\boldsymbol{\omega}_{\text{ref}}^b = \mathbf{R}_I^b (\mathbf{x}_{\text{ref}}^I \times \dot{\mathbf{x}}_{\text{ref}}^I) = \mathbf{x}_{\text{ref}}^b \times (\mathbf{R}_I^b \dot{\mathbf{x}}_{\text{ref}}^I) \quad (6.6)$$

where $\mathbf{R}_I^b = \mathbf{R}_b^{I^\top}$, and $\mathbf{R}_I^I$ describes the full orientation of the body frame b .

As in [124], the control law will be realized in the body frame, in which the directions of the control inputs are fixed. The control objective is then to align the body-frame representation $\mathbf{x}_{\text{ref}}^b = \mathbf{R}_I^b \mathbf{x}_{\text{ref}}^I$ of the reference with the unactuated axis $\mathbf{e}_1$ of the body frame. The shortest distance between the two vectors along the surface of the unit sphere $\mathbb{S}^2$ is equal to the angle $\tilde{\phi} \in [0, \pi]$ between them, given as

$$\tilde{\phi} = \arccos(\mathbf{e}_1^\top \mathbf{x}_{\text{ref}}^b) . \quad (6.7)$$

Let $\tilde{\mathbf{R}}$ be the rotation matrix rotating $\mathbf{x}_{\text{ref}}^b$ to $\mathbf{e}_1$, such that $\tilde{\mathbf{R}}\mathbf{x}_{\text{ref}}^b = \mathbf{e}_1$. The rotation described by $\tilde{\mathbf{R}}$ is a rotation about the axis parallel to $\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1$ by an angle $\tilde{\phi}$. Using the fact that $\|\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1\| = |\sin \tilde{\phi}| = \sin \tilde{\phi}$ for $\mathbf{x}_{\text{ref}}^b \neq -\mathbf{e}_1$, $\tilde{\mathbf{R}}$ can be expressed as [46]

$$\tilde{\mathbf{R}} = \mathbf{I}_3 + \mathbf{S}(\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) + \left(\frac{1 - \cos \tilde{\phi}}{\sin^2 \tilde{\phi}} \right) \mathbf{S}^2(\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) . \quad (6.8)$$

The matrix $\tilde{\mathbf{R}}^\top$ can be interpreted as describing the orientation of another frame relative to the body frame. The first axis of this frame is aligned with the reference $\mathbf{x}_{\text{ref}}^b$, and the velocity of this frame relative to the body frame, given in the body frame is $-(\boldsymbol{\omega}^b - \boldsymbol{\omega}_{\text{ref}}^b)$. Consequently,

$$\dot{\mathbf{x}}_{\text{ref}}^b = \dot{\tilde{\mathbf{R}}}^\top \mathbf{e}_1 = \mathbf{S}(-(\boldsymbol{\omega}^b - \boldsymbol{\omega}_{\text{ref}}^b)) \tilde{\mathbf{R}}^\top \mathbf{e}_1 = -(\boldsymbol{\omega}^b - \boldsymbol{\omega}_{\text{ref}}^b) \times \mathbf{x}_{\text{ref}}^b . \quad (6.9)$$

The difference $\boldsymbol{\omega}^b - \boldsymbol{\omega}_{\text{ref}}^b$ between the angular velocities is unsuited to use for feedback, as it cannot be realized by the available control inputs. This problem is addressed by [23], where the matrix $\tilde{\mathbf{R}}$ is used to rotate the angular velocity of the reference, which lies in a plane tangential to the unit sphere at the point $\mathbf{x}_{\text{ref}}^b$, into the tangent plane at $\mathbf{e}_1$, which is also the plane spanned by the two unactuated axes. This is illustrated in Figure 6.1. Rotating the reference angular velocity allows us to formulate an angular velocity error $\tilde{\boldsymbol{\omega}}^b \in \mathbb{R}^2$ which can be fully realized by the available control inputs, given as

$$\tilde{\boldsymbol{\omega}}^b = \mathbf{H}^\top (\boldsymbol{\omega}^b - \tilde{\mathbf{R}} \boldsymbol{\omega}_{\text{ref}}^b) . \quad (6.10)$$

Multiplication by $\mathbf{H}^\top$ disregards only the angular velocity of the body about the first axis of the body frame, which does not influence its pointing direction.

In order to relate the difference $\boldsymbol{\omega}^b - \boldsymbol{\omega}_{\text{ref}}^b$ to the error $\tilde{\boldsymbol{\omega}}^b$, we first decompose $\boldsymbol{\omega}^b$ into a component parallel to $\mathbf{e}_1$ and a component perpendicular to $\mathbf{e}_1$, i.e. which lies in the plane spanned by the actuated axes of the body frame:

$$\boldsymbol{\omega}^b = \underbrace{\mathbf{e}_1 \mathbf{e}_1^\top \boldsymbol{\omega}^b}_{=\boldsymbol{\omega}^{b\parallel}} + \mathbf{H} \mathbf{H}^\top \boldsymbol{\omega}^b , \quad (6.11)$$

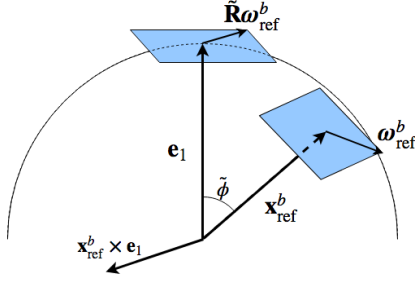


Figure 6.1: Illustration of the reference velocity ω_{ref}^b being rotated by $\tilde{R}$.

where $\mathbf{H}\mathbf{H}^\top$ is a projection matrix which projects a vector into the plane spanned by the actuated axes. We can then write

$$\begin{aligned}\omega^b - \omega_{\text{ref}}^b &= \mathbf{H}\mathbf{H}^\top \omega^b + \omega^{\text{b}\parallel} - \tilde{R}\omega_{\text{ref}}^b + (\tilde{R} - \mathbf{I}_3)\omega_{\text{ref}}^b \\ &= \mathbf{H}\tilde{\omega}^b + \omega^{\text{b}\parallel} + (\tilde{R} - \mathbf{I}_3)\omega_{\text{ref}}^b\end{aligned}\quad (6.12)$$

where the first term is perpendicular to $\mathbf{e}_1$, the second parallel to $\mathbf{e}_1$, and from the expression (6.8) for $\tilde{R}$, it follows that the third term of (6.12) is perpendicular to $\mathbf{e}_1 \times \mathbf{x}_{\text{ref}}^b$.

6.2.3 Control law and analysis of closed-loop dynamics

To achieve the control objective, we propose the following control law:

$$\mathbf{u} = -k_P \mathbf{H}^\top \chi(\mathbf{x}_{\text{ref}}^b) - k_D \tilde{\omega}^b + \mathbf{H}^\top \left(J \tilde{R} \mathbf{R}_I^b (\mathbf{x}_{\text{ref}}^I \times \ddot{\mathbf{x}}_{\text{ref}}^I) - \mathbf{m} \right) \quad (6.13)$$

where $\chi(\mathbf{x}_{\text{ref}}^b)$ is an error function we define to be

$$\chi(\mathbf{x}_{\text{ref}}^b) = \frac{\tilde{\phi}}{\|\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1\|} (\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) = \frac{\arccos(\mathbf{e}_1^\top \mathbf{x}_{\text{ref}}^b)}{\sin(\arccos(\mathbf{e}_1^\top \mathbf{x}_{\text{ref}}^b))} (\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1). \quad (6.14)$$

The error χ is defined for all $\mathbf{x}_{\text{ref}}^b$ on $\mathbb{S}^2 \setminus \{-\mathbf{e}_1\}$. The resulting proportional action term is a torque about the axis $\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1$. Its magnitude is $k_P \tilde{\phi}$, as in [23]. The control law (6.13) is inspired by the works [23] and [124], but unlike the tracking law proposed by [23], the inputs are torques, requiring the inertia of the system to be taken into account, and unlike in [124], the system (6.4) considered here is underactuated.

When $\mathbf{x}_{\text{ref}}^b = -\mathbf{e}_1$, the matrix $\tilde{R}$ is not uniquely defined, and the error function χ is also undefined. At this point, the antipodal point, they are set to be $\tilde{R} = \mathbf{I}_3$ and $\chi(-\mathbf{e}_1) = \mathbf{0}$, leading to a discontinuity in the control law.

Remark. In order to avoid the discontinuity due to $\tilde{R}$ not being uniquely defined at the antipodal point, the reference velocity could be projected into the tangent plane of $\mathbf{e}_1$ rather than rotated, as has later been done in [35].

The dynamics of the angular velocity error $\tilde{\omega}^b$ are

$$\begin{aligned}
 \dot{\tilde{\omega}}^b &= \mathbf{H}^\top \left(\dot{\omega}^b - \dot{\tilde{\mathbf{R}}} \omega_{\text{ref}}^b - \tilde{\mathbf{R}} \dot{\omega}_{\text{ref}}^b \right) \\
 &= \mathbf{H}^\top \left(\mathbf{J}^{-1} (\mathbf{m} + \mathbf{H}\mathbf{u}) - \tilde{\mathbf{R}} \left((\omega^b - \omega_{\text{ref}}^b) \times \omega_{\text{ref}}^b \right) \right. \\
 &\quad \left. - \tilde{\mathbf{R}} (-\omega^b \times \omega_{\text{ref}}^b + \mathbf{R}_I^b (\mathbf{x}_{\text{ref}}^I \times \ddot{\mathbf{x}}_{\text{ref}}^I)) \right) \\
 &= \mathbf{H}^\top \left(\mathbf{J}^{-1} (\mathbf{m} + \mathbf{H}\mathbf{u}) - \tilde{\mathbf{R}} \mathbf{R}_I^b (\mathbf{x}_{\text{ref}}^I \times \ddot{\mathbf{x}}_{\text{ref}}^I) \right)
 \end{aligned} \tag{6.15}$$

The matrix $\mathbf{H}\mathbf{H}^\top$ is a projection onto the plane tangent to $\mathbf{e}_1$, and therefore for any vector $\mathbf{v} \perp \mathbf{e}_1$ it holds that $\mathbf{H}\mathbf{H}^\top \mathbf{v} = \mathbf{v}$. Furthermore, the block-diagonal structure of $\mathbf{J}$ results in $\mathbf{J}^{-1} \mathbf{H}\mathbf{H}^\top \mathbf{J} = \mathbf{H}\mathbf{H}^\top$. After rotation by $\tilde{\mathbf{R}}$, the angular acceleration term $\mathbf{R}_I^b (\mathbf{x}_{\text{ref}}^I \times \ddot{\mathbf{x}}_{\text{ref}}^I)$ is perpendicular to $\mathbf{e}_1$, and can therefore be fully cancelled by the chosen input. Inserting the control law (6.13) for $\mathbf{u}$, and using the fact that $\mathbf{H}^\top \mathbf{J}^{-1} \mathbf{H} = \mathbf{J}_{22}^{-1}$, gives

$$\dot{\tilde{\omega}}^b = \mathbf{J}_{22}^{-1} (-k_P \mathbf{H}^\top \boldsymbol{\chi} - k_D \tilde{\omega}^b) + \mathbf{H}^\top \mathbf{J}^{-1} (\mathbf{I}_3 - \mathbf{H}\mathbf{H}^\top) \mathbf{m} \tag{6.16}$$

The block-diagonal structure of $\mathbf{J}$, and consequently $\mathbf{J}^{-1}$, means that the term $\mathbf{J}^{-1} (\mathbf{I}_3 - \mathbf{H}\mathbf{H}^\top) \mathbf{m}$ is parallel to $\mathbf{e}_1$, and therefore disappears upon multiplication with $\mathbf{H}^\top$. In other words, while the torque $\mathbf{m}$ is only partially cancelled by the control input, what is not cancelled is only the torque about the unactuated axis, which does not influence its pointing direction. The closed-loop error dynamics of the angular velocity are then

$$\dot{\tilde{\omega}}^b = -k_P \mathbf{J}_{22}^{-1} \mathbf{H}^\top \boldsymbol{\chi} - k_D \mathbf{J}_{22}^{-1} \tilde{\omega}^b. \tag{6.17}$$

Theorem 6.1. *For a system with dynamics (6.4) and subject to Assumption 6.1, the control law (6.13) makes the solution $(\mathbf{x}_{\text{ref}}^b(t), \tilde{\omega}^b(t)) = (\mathbf{e}_1, \mathbf{0})$ of the closed-loop system dynamics (6.9), (6.17) US for initial conditions $(\mathbf{x}_{\text{ref}}^b(t_0), \tilde{\omega}^b(t_0))$ such that*

$$\tilde{\phi}(t_0)^2 + \frac{1}{k_P} \tilde{\omega}^b(t_0)^\top \mathbf{J}_{22} \tilde{\omega}^b(t_0) < \pi^2 \tag{6.18}$$

where $\tilde{\phi}(t_0) = \arccos(\mathbf{e}_1^\top \mathbf{x}_{\text{ref}}^b(t_0))$. Furthermore, if the first element of ω^b , i.e. $\omega^b_{\parallel}(t)$, is bounded, the tracking errors $(\tilde{\phi}, \tilde{\omega}^b)$ converge to zero exponentially.

Remark. The assumption that $\omega^b_{\parallel}(t)$ is bounded needs to be verified for every particular system dynamics in closed loop with the input (6.13).

Proof of Theorem 6.1. The proof consists of the following two main steps: First, uniform stability of the solution is proved, and the region of initial conditions for which this holds is established. Then, exponential convergence of the errors is proved.

Consider the domain $\mathbb{S}^2 \setminus \{-\mathbf{e}_1\} \times \mathbb{R}^2$, denoted D , and a Lyapunov function candidate $V_1(\mathbf{x}_{\text{ref}}^b, \tilde{\omega}^b) : D \rightarrow \mathbb{R}$ chosen as

$$V_1(\mathbf{x}_{\text{ref}}^b, \tilde{\omega}^b) = \frac{k_P}{2} \|\boldsymbol{\chi}(\mathbf{x}_{\text{ref}}^b)\|^2 + \frac{1}{2} \tilde{\omega}^{b\top} \mathbf{J}_{22} \tilde{\omega}^b \tag{6.19}$$

which is a continuously differentiable function on the domain D and which satisfies $V_1(\mathbf{e}_1, \mathbf{0}) = 0$ and $V_1(\mathbf{x}_{\text{ref}}^b, \tilde{\boldsymbol{\omega}}^b) > 0 \forall (\mathbf{x}_{\text{ref}}^b, \tilde{\boldsymbol{\omega}}^b) \in D \setminus (\mathbf{e}_1, \mathbf{0})$. Differentiating V_1 and inserting the system dynamics (6.9), (6.17) gives

$$\begin{aligned}
\dot{V}_1 &= \frac{k_P}{2} \frac{d}{dt} \left((\arccos(\mathbf{e}_1^\top \mathbf{x}_{\text{ref}}^b))^2 \right) + \tilde{\boldsymbol{\omega}}^{b\top} \mathbf{J}_{22} \dot{\tilde{\boldsymbol{\omega}}}^b \\
&= k_P \arccos(\mathbf{e}_1^\top \mathbf{x}_{\text{ref}}^b) \frac{-1}{\sqrt{1 - (\mathbf{e}_1^\top \mathbf{x}_{\text{ref}}^b)^2}} \mathbf{e}_1^\top \dot{\mathbf{x}}_{\text{ref}}^b + \tilde{\boldsymbol{\omega}}^{b\top} \mathbf{J}_{22} \dot{\tilde{\boldsymbol{\omega}}}^b \\
&= -k_P \frac{\tilde{\phi}}{\sin \tilde{\phi}} \mathbf{e}_1^\top (-(\boldsymbol{\omega}^b - \boldsymbol{\omega}_{\text{ref}}^b) \times \mathbf{x}_{\text{ref}}^b) - k_D \|\tilde{\boldsymbol{\omega}}^b\|^2 - k_P \tilde{\boldsymbol{\omega}}^{b\top} \mathbf{H}^\top \boldsymbol{\chi} \\
&= k_P \frac{\tilde{\phi}}{\sin \tilde{\phi}} (\boldsymbol{\omega}^b - \boldsymbol{\omega}_{\text{ref}}^b)^\top (\mathbf{e}_1 \times \mathbf{x}_{\text{ref}}^b) - k_D \|\tilde{\boldsymbol{\omega}}^b\|^2 - k_P \tilde{\boldsymbol{\omega}}^{b\top} \mathbf{H}^\top \boldsymbol{\chi} \\
&= k_P \boldsymbol{\chi}^\top \mathbf{H} \tilde{\boldsymbol{\omega}}^b - k_D \|\tilde{\boldsymbol{\omega}}^b\|^2 - k_P \tilde{\boldsymbol{\omega}}^{b\top} \mathbf{H}^\top \boldsymbol{\chi} \\
&= -k_D \|\tilde{\boldsymbol{\omega}}^b\|^2 \leq 0
\end{aligned} \tag{6.20}$$

where (6.3) has been used, and (6.12) inserted for $\boldsymbol{\omega}^b - \boldsymbol{\omega}_{\text{ref}}^b$. Hence by [78, Theorem 4.8], the solution $(\mathbf{x}_{\text{ref}}^b(t), \tilde{\boldsymbol{\omega}}^b(t)) \equiv (\mathbf{e}_1, 0)$ is US.

Furthermore, since $V \leq 0$ and thus $V_1(t) \leq V_1(t_0)$ for $t \geq t_0$, we have

$$|\tilde{\phi}(t)|^2 \leq \frac{2}{k_P} V_1(t) \leq \frac{2}{k_P} V_1(t_0) = \tilde{\phi}(t_0)^2 + \frac{1}{k_P} \tilde{\boldsymbol{\omega}}^b(t_0)^\top \mathbf{J}_{22} \tilde{\boldsymbol{\omega}}^b(t_0). \tag{6.21}$$

Thus for any solution starting from initial conditions satisfying (6.18), $|\tilde{\phi}(t)| < \pi \forall t \geq t_0$, meaning that the solutions stay within the domain D on which $\boldsymbol{\chi}$ and V_1 are defined.

Consider now a second Lyapunov function candidate $V_2(\mathbf{x}_{\text{ref}}^b, \tilde{\boldsymbol{\omega}}^b) : D \rightarrow \mathbb{R}$:

$$V_2(\mathbf{x}_{\text{ref}}^b, \tilde{\boldsymbol{\omega}}^b) = V_1 + \varepsilon \underbrace{\boldsymbol{\chi}^\top \mathbf{H} \tilde{\boldsymbol{\omega}}^b}_{=W} = \frac{1}{2} \begin{bmatrix} \boldsymbol{\chi}^\top & \tilde{\boldsymbol{\omega}}^{b\top} \end{bmatrix} \underbrace{\begin{bmatrix} k_P \mathbf{I}_3 & \varepsilon \mathbf{H} \\ \varepsilon \mathbf{H}^\top & \mathbf{J}_{22} \end{bmatrix}}_{=\mathbf{P}} \begin{bmatrix} \boldsymbol{\chi} \\ \tilde{\boldsymbol{\omega}}^b \end{bmatrix} \tag{6.22}$$

where the matrix $\mathbf{P}$ is positive definite for sufficiently small $\varepsilon > 0$, specifically $\varepsilon^2 < k_P \lambda_{\min}(\mathbf{J}_{22})$. Furthermore, V_2 can be bounded by

$$\lambda_{\min}(\mathbf{P}) (\|\boldsymbol{\chi}\|^2 + \|\tilde{\boldsymbol{\omega}}^b\|^2) \leq V_2 \leq \lambda_{\max}(\mathbf{P}) (\|\boldsymbol{\chi}\|^2 + \|\tilde{\boldsymbol{\omega}}^b\|^2). \tag{6.23}$$

The derivative of W is

$$\dot{W} = \dot{\boldsymbol{\chi}}^\top \mathbf{H} \tilde{\boldsymbol{\omega}}^b + \boldsymbol{\chi}^\top \mathbf{H} \dot{\tilde{\boldsymbol{\omega}}}^b. \tag{6.24}$$

To find $\dot{\boldsymbol{\chi}}$, we differentiate (6.14) and obtain

$$\begin{aligned}
\dot{\boldsymbol{\chi}} &= \frac{\tilde{\phi} \cos \tilde{\phi} - \sin \tilde{\phi}}{\sin^3 \tilde{\phi}} (\mathbf{e}_1^\top \dot{\mathbf{x}}_{\text{ref}}^b) (\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) + \frac{\tilde{\phi}}{\sin \tilde{\phi}} (\dot{\mathbf{x}}_{\text{ref}}^b \times \mathbf{e}_1) \\
&= \frac{\tilde{\phi} \cos \tilde{\phi} - \sin \tilde{\phi}}{\sin^3 \tilde{\phi}} ((-\mathbf{e}_1^\top (\boldsymbol{\omega}^b - \boldsymbol{\omega}_{\text{ref}}^b) \times \mathbf{x}_{\text{ref}}^b)) (\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) \\
&\quad + \frac{\tilde{\phi}}{\sin \tilde{\phi}} (- (\boldsymbol{\omega}^b - \boldsymbol{\omega}_{\text{ref}}^b) \times \mathbf{x}_{\text{ref}}^b) \times \mathbf{e}_1.
\end{aligned} \tag{6.25}$$

Inserting (6.12) for $\omega^b - \omega_{\text{ref}}^b$, and applying (6.3) to the first term of (6.25) yields

$$\begin{aligned}
 \dot{\chi} &= \frac{\sin \tilde{\phi} - \tilde{\phi} \cos \tilde{\phi}}{\sin^3 \tilde{\phi}} \left(\left(\mathbf{H} \tilde{\omega}^b + \omega^{b\parallel} + \underbrace{(\tilde{\mathbf{R}} - \mathbf{I}_3) \omega_{\text{ref}}^b}_{\perp \mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1} \right)^\top (\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) \right) (\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) \\
 &\quad + \frac{\tilde{\phi}}{\sin \tilde{\phi}} \left(- \left(\mathbf{H} \tilde{\omega}^b + \omega^{b\parallel} + (\tilde{\mathbf{R}} - \mathbf{I}_3) \omega_{\text{ref}}^b \right) \times \mathbf{x}_{\text{ref}}^b \right) \times \mathbf{e}_1 \\
 &= \frac{\sin \tilde{\phi} - \tilde{\phi} \cos \tilde{\phi}}{\sin^3 \tilde{\phi}} \left((\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1)^\top \mathbf{H} \tilde{\omega}^b \right) (\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) \\
 &\quad - \underbrace{\frac{\tilde{\phi}}{\sin \tilde{\phi}} \left((\mathbf{H} \tilde{\omega}^b + \omega^{b\parallel}) \times \mathbf{x}_{\text{ref}}^b \right) \times \mathbf{e}_1}_{=\mathbf{a}} \\
 &\quad - \frac{\tilde{\phi}}{\sin \tilde{\phi}} \left(\left((\tilde{\mathbf{R}} - \mathbf{I}_3) \omega_{\text{ref}}^b \right) \times \mathbf{x}_{\text{ref}}^b \right) \times \mathbf{e}_1.
 \end{aligned} \tag{6.26}$$

Consider now just the term $\mathbf{a}$. Applying (6.2) and then (6.1) gives

$$\begin{aligned}
 \mathbf{a} &= - \frac{\tilde{\phi}}{\sin \tilde{\phi}} \left((\mathbf{H} \tilde{\omega}^b + \omega^{b\parallel}) \times (\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) + \mathbf{x}_{\text{ref}}^b \times ((\mathbf{H} \tilde{\omega}^b + \omega^{b\parallel}) \times \mathbf{e}_1) \right) \\
 &= - (\mathbf{H} \tilde{\omega}^b + \omega^{b\parallel}) \times \chi + \frac{\tilde{\phi}}{\sin \tilde{\phi}} \left((\mathbf{x}_{\text{ref}}^{b\top} \mathbf{e}_1) \mathbf{H} \tilde{\omega}^b - (\mathbf{x}_{\text{ref}}^{b\top} \mathbf{H} \tilde{\omega}^b) \mathbf{e}_1 \right).
 \end{aligned} \tag{6.27}$$

We can now return to $\dot{W}$ given by (6.24). Inserting (6.26), (6.27) for $\dot{\chi}$, ignoring terms which are perpendicular to $\mathbf{H} \tilde{\omega}^b$ and therefore vanish in the inner product, gives

$$\begin{aligned}
 \dot{W} &= \dot{\chi}^\top \mathbf{H} \tilde{\omega}^b + \chi^\top \mathbf{H} \dot{\tilde{\omega}}^b \\
 &= \frac{\sin \tilde{\phi} - \tilde{\phi} \cos \tilde{\phi}}{\sin^3 \tilde{\phi}} \left((\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1)^\top \mathbf{H} \tilde{\omega}^b \right)^2 \\
 &\quad - (\omega^{b\parallel} \times \chi)^\top \mathbf{H} \tilde{\omega}^b + \frac{\tilde{\phi} \cos \tilde{\phi}}{\sin \tilde{\phi}} (\mathbf{H} \tilde{\omega}^b)^\top \mathbf{H} \tilde{\omega}^b \\
 &\quad - \frac{\tilde{\phi}}{\sin \tilde{\phi}} \left(\left(((\tilde{\mathbf{R}} - \mathbf{I}_3) \omega_{\text{ref}}^b) \times \mathbf{x}_{\text{ref}}^b \right) \times \mathbf{e}_1 \right)^\top \mathbf{H} \tilde{\omega}^b \\
 &\quad + \chi^\top \mathbf{H} (-k_P \mathbf{J}_{22}^{-1} \mathbf{H}^\top \chi - k_D \mathbf{J}_{22}^{-1} \tilde{\omega}^b).
 \end{aligned} \tag{6.28}$$

Now let the angle between χ and $\mathbf{H} \tilde{\omega}^b$ be denoted ψ . We then have

$$\begin{aligned}
 \dot{W} &= \left(\cos^2 \psi + (1 - \cos^2 \psi) \frac{\tilde{\phi} \cos \tilde{\phi}}{\sin \tilde{\phi}} \right) \|\tilde{\omega}^b\|^2 - (\omega^{b\parallel} \times \chi)^\top \mathbf{H} \tilde{\omega}^b \\
 &\quad - \frac{\tilde{\phi}}{\sin \tilde{\phi}} \left(\left(((\tilde{\mathbf{R}} - \mathbf{I}_3) \omega_{\text{ref}}^b) \times \mathbf{x}_{\text{ref}}^b \right) \times \mathbf{e}_1 \right)^\top \mathbf{H} \tilde{\omega}^b \\
 &\quad - k_P \chi^\top \mathbf{H} \mathbf{J}_{22}^{-1} \mathbf{H}^\top \chi - k_D \chi^\top \mathbf{H} \mathbf{J}_{22}^{-1} \tilde{\omega}^b \\
 &\leq \left(1 + \frac{|\tilde{\phi} \cos \tilde{\phi}|}{\sin \tilde{\phi}} \right) \|\tilde{\omega}^b\|^2 + \tilde{\phi} \|\omega^{b\parallel}\| \cdot \|\tilde{\omega}^b\| + \frac{\tilde{\phi}}{\sin \tilde{\phi}} \left\| (\tilde{\mathbf{R}} - \mathbf{I}_3) \omega_{\text{ref}}^b \right\| \cdot \|\tilde{\omega}^b\| \\
 &\quad - k_P \lambda_{\min}(\mathbf{J}_{22}^{-1}) \tilde{\phi}^2 + k_D \lambda_{\max}(\mathbf{J}_{22}^{-1}) \tilde{\phi} \|\tilde{\omega}^b\|
 \end{aligned} \tag{6.29}$$

where we have used that $\|\mathbf{H}\boldsymbol{\chi}\| = \|\boldsymbol{\chi}\|$, since $\boldsymbol{\chi} \perp \mathbf{e}_1$. To evaluate the term $\|(\tilde{\mathbf{R}} - \mathbf{I}_3)\boldsymbol{\omega}_{\text{ref}}^b\|$, we insert (6.8) for $\tilde{\mathbf{R}}$, which gives

$$\begin{aligned} & \left\| (\tilde{\mathbf{R}} - \mathbf{I}_3)\boldsymbol{\omega}_{\text{ref}}^b \right\| = \\ & \left\| (\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) \times \boldsymbol{\omega}_{\text{ref}}^b + \frac{1 - \cos \tilde{\phi}}{\sin^2 \tilde{\phi}} (\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) \times ((\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1) \times \boldsymbol{\omega}_{\text{ref}}^b) \right\| \\ & \leq \|\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1\| \cdot \|\boldsymbol{\omega}_{\text{ref}}^b\| + \frac{|1 - \cos \tilde{\phi}|}{\sin^2 \tilde{\phi}} \|\mathbf{x}_{\text{ref}}^b \times \mathbf{e}_1\|^2 \cdot \|\boldsymbol{\omega}_{\text{ref}}^b\| \\ & = (\sin \tilde{\phi} + |1 - \cos \tilde{\phi}|) \|\boldsymbol{\omega}_{\text{ref}}^b\|. \end{aligned} \quad (6.30)$$

Inserting this into (6.29) gives the following bound on $\dot{W}$:

$$\begin{aligned} \dot{W} \leq & \left(1 + \frac{|\tilde{\phi} \cos \tilde{\phi}|}{\sin \tilde{\phi}}\right) \|\tilde{\boldsymbol{\omega}}^b\|^2 + \left(1 + \frac{|1 - \cos \tilde{\phi}|}{\sin \tilde{\phi}}\right) \tilde{\phi} \|\boldsymbol{\omega}_{\text{ref}}^b\| \cdot \|\tilde{\boldsymbol{\omega}}^b\| \\ & + \tilde{\phi} \|\boldsymbol{\omega}^{b\parallel}\| \cdot \|\tilde{\boldsymbol{\omega}}^b\| - k_P \frac{1}{\lambda_{\max}(\mathbf{J}_{22})} \tilde{\phi}^2 + k_D \frac{1}{\lambda_{\min}(\mathbf{J}_{22})} \tilde{\phi} \|\tilde{\boldsymbol{\omega}}^b\|. \end{aligned} \quad (6.31)$$

Since $\tilde{\phi}$ is bounded away from π , there exist constants $M_1, M_2 > 0$ s.t.

$$\frac{|\tilde{\phi} \cos \tilde{\phi}|}{\sin \tilde{\phi}} \leq M_1, \quad \frac{|1 - \cos \tilde{\phi}|}{\sin \tilde{\phi}} \leq M_2. \quad (6.32)$$

Using the bound (6.31) and inserting (6.20) for $\dot{V}_1$, $\dot{V}_2$ can be bounded as follows:

$$\begin{aligned} \dot{V}_2 = \dot{V}_1 + \varepsilon \dot{W} \\ \leq - \begin{bmatrix} \|\boldsymbol{\chi}\| & \|\tilde{\boldsymbol{\omega}}^b\| \end{bmatrix} \underbrace{\begin{bmatrix} \varepsilon \frac{k_P}{\lambda_{\max}(\mathbf{J}_{22})} & -\frac{\varepsilon}{2} M_3 \\ -\frac{\varepsilon}{2} M_3 & k_D - \varepsilon(1 + M_1) \end{bmatrix}}_{=\mathbf{Q}} \begin{bmatrix} \|\boldsymbol{\chi}\| \\ \|\tilde{\boldsymbol{\omega}}^b\| \end{bmatrix} \end{aligned} \quad (6.33)$$

where

$$M_3 = \|\boldsymbol{\omega}^{b\parallel}\| + (1 + M_2) \|\boldsymbol{\omega}_{\text{ref}}^b\| + \frac{k_D}{\lambda_{\min}(\mathbf{J}_{22})}. \quad (6.34)$$

Provided that $\|\boldsymbol{\omega}^{b\parallel}\|$ and $\|\boldsymbol{\omega}_{\text{ref}}^b\|$ bounded, one can choose ε sufficiently small so that the matrix $\mathbf{Q}$ is positive definite. Consequently,

$$\dot{V}_2 \leq -\lambda_{\min}(\mathbf{Q}) (\|\boldsymbol{\chi}\|^2 + \|\tilde{\boldsymbol{\omega}}^b\|^2) \leq -\frac{\lambda_{\min}(\mathbf{Q})}{\lambda_{\max}(\mathbf{P})} V_2. \quad (6.35)$$

By the comparison lemma [78, Lemma 3.4],

$$V_2(\mathbf{x}_{\text{ref}}^b(t), \tilde{\boldsymbol{\omega}}^b(t)) \leq V_2(\mathbf{x}_{\text{ref}}^b(t_0), \tilde{\boldsymbol{\omega}}^b(t_0)) e^{-\frac{\lambda_{\min}(\mathbf{Q})}{\lambda_{\max}(\mathbf{P})}(t-t_0)} \quad (6.36)$$

and consequently

$$\left\| \begin{bmatrix} \|\boldsymbol{\chi}(t)\| \\ \|\tilde{\boldsymbol{\omega}}^b(t)\| \end{bmatrix} \right\| \leq \sqrt{\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}} \left\| \begin{bmatrix} \|\boldsymbol{\chi}(t_0)\| \\ \|\tilde{\boldsymbol{\omega}}^b(t_0)\| \end{bmatrix} \right\| e^{-\frac{\lambda_{\min}(\mathbf{Q})}{2\lambda_{\max}(\mathbf{P})}(t-t_0)}, \quad (6.37)$$

meaning that the errors $\tilde{\phi} = \|\boldsymbol{\chi}\|$ and $\tilde{\boldsymbol{\omega}}^b$ go exponentially to zero. $\square$

6.3 Path following

In this section, the attitude control law developed in Section 6.2 is used to provide not control torques, but references for the joints of an AIAUV with dynamics given in Section 2.1.1. This way, the pointing direction of the AIAUV is controlled indirectly by curving its body. This is combined with a guidance law to achieve path following of straight paths in 3D.

6.3.1 Control objective

The task to be solved is for the front of the head link, where an end-effector or camera typically is mounted, to converge towards and follow a straight path in a 3-dimensional space. The path is chosen to be parallel to the x -axis of an inertial reference frame. The control objective of following the path is then equivalent to driving the y - and z -positions, p_y , p_z , of the end-effector in the inertial frame to 0, while maintaining a desired constant forward velocity u_d , which can be formalized as

$$\lim_{t \rightarrow \infty} p_y = 0, \quad \lim_{t \rightarrow \infty} p_z = 0, \quad \lim_{t \rightarrow \infty} u = u_d \quad (6.38)$$

where u denotes the surge velocity of the end-effector.

6.3.2 Guidance

The following guidance law is a vector formulation of the LOS principle. It is obtained by extending the guidance law for surface vessels from [119] to 3D. The desired direction of travel $\boldsymbol{\mu}_{\text{ref}}^I$ is then

$$\boldsymbol{\mu}_{\text{ref}}^I = u_d \mathbf{e}_1 - k_e [0, p_y, p_z]^\top \quad (6.39)$$

where u_d is the desired forward velocity, and k_e is a design parameter. The guidance law is illustrated in Figure 6.2. The parameter k_e adjusts the convergence rate towards the path. Different weighting of the errors p_y , p_z , achieving different convergence rates, can be obtained by replacing the scalar k_e with a diagonal matrix.

Remark. The guidance law (6.39) is very similar to the one also used in [156], and differs only in that it is the cross-track error rather than the component along the path that is weighted by a design parameter. Fixing the along-path component to have the same magnitude as the desired forward velocity makes it easier to combine the guidance law with a term taking into account the presence of a current as in [119], since the ratio between the current compensating term and along-path term must match the ratio between the magnitudes of the current and the desired velocity.

6.3.3 Joint angle references for direction control

The AIAUVs considered in this chapter are assumed to have an equal number of joints actuated about the y - and z -axes of the link frames to which they are attached. The joints will be used for direction control, by curving the body of the AIAUV to achieve a turning motion. In order to keep the pointing direction of the end-effector

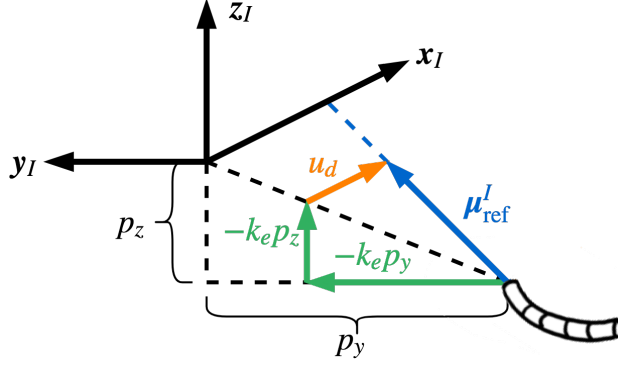


Figure 6.2: Illustration of the guidance law (6.39). The axes of the inertial frame are denoted x_I , y_I , z_I .

steady, it is desirable to use the joints towards the back of the AIAUV more than those in the front, near the end-effector. In [135], the joint references increase linearly towards the tail. In this section, we propose an exponential scaling factor for the joint references, inspired by the head-to-tail amplitude ratio of swimming biological eels [122].

Let $\theta_{a,j}$, $j = 1, \dots, \frac{n-1}{2}$ denote the j th joint among the joints actuated about axis $a = y, z$, numbered from tail to head. The reference angle $\theta_{a,j}^*$ is chosen to be

$$\theta_{a,j}^* = e^{-\alpha(j-1)} \text{sat}(\theta_{a,0}) \quad (6.40)$$

where $\theta_{a,0}$ is a base angle common for all joints rotating about the same axis, to be specified later. The $\text{sat}(\cdot)$ -function is used to saturate the reference at a maximum angle $\theta_{\max}$ in order to preserve the shape of the curve despite physical joint limitations. The shape of the curve can be adjusted by choice of the parameter $\alpha > 0$. Larger values of α will result in keeping the front of the AIAUV more straight and using primarily the joints towards the back to turn, reminiscent of the tail of a fish or the rudder of a traditional marine vehicle, whereas a lower value of α will result in all joints being used more equally, giving a more uniform curve of the body.

Now let $\boldsymbol{\theta}_0 = [\theta_{y,0}, \theta_{z,0}]^\top$ be a vector consisting of the base reference angles for the joints actuated about the y - and z -axes, respectively. Based on the control law (6.13), these are chosen to be

$$\boldsymbol{\theta}_0 = -k_P \mathbf{H}^\top \boldsymbol{\chi}(\mathbf{x}_{\text{ref}}^b) - k_D \tilde{\boldsymbol{\omega}}^e + \mathbf{H}^\top \tilde{\mathbf{R}} \mathbf{R}_I^e (\mathbf{x}_{\text{ref}}^I \times \ddot{\mathbf{x}}_{\text{ref}}^I) \quad (6.41)$$

where e denotes the end-effector frame. The angle $\tilde{\phi}$ is the angle between the end-effector and inertial frame x -axes. The reference pointing direction $\mathbf{x}_{\text{ref}}^I$ is the unit vector parallel to $\boldsymbol{\mu}_{\text{ref}}^I$ from the guidance law (6.39). Since the base angles influence the pointing direction of the AIAUV indirectly, the last term of (6.41) serves to incorporate some information about the future motion of the reference, but will not necessarily ensure tracking.

As in Section 6.2, the term $\boldsymbol{\chi}$ is set to zero when the reference and current pointing direction are parallel but pointing in opposite directions. However, the

AIAUV can only remain pointing opposite its reference if it is travelling along the path in the opposite direction with straightened joints.

6.3.4 Velocity and joint control

The controllers described in this section will be applied to the dynamic model of the AIAUV described in Section 2.1.1.

Joints

As in [149], the joints will be controlled using a PD-controller, but with an additional term compensating for the hydrostatic forces acting on the individual joints. The i th element $\tau_{\theta,i}$ of the control input $\boldsymbol{\tau}_\theta$ is set to

$$\tau_{\theta,i} = -k_{\theta,P}(\theta_i - \theta_i^*) - k_{\theta,D}\dot{\theta}_i + g_{6+i} \quad (6.42)$$

where θ_i^* denotes the reference for joint i among all joints. The term g_{6+i} denotes the $6+i$ th element of the hydrostatic term $\mathbf{g}(\mathbf{q}, \boldsymbol{\theta})$ in (2.1b), which corresponds to the hydrostatic moment acting on joint i , which influences the joints. In order to avoid differentiating the joint reference (6.40), the controller (6.42) effectively treats the reference as a setpoint.

Forward velocity

The forward velocity u of the end-effector will be controlled using only the longitudinal thrusters of the AIAUV. The i th element $\tau_{\text{thr},i}$ of $\boldsymbol{\tau}_{\text{thr}}$ is set to

$$\tau_{\text{thr},i} = -k_{1,P}(u - u_d) - k_{1,I} \int_{t_0}^t u(\tau) - u_d \, d\tau \quad (6.43)$$

if thruster i acts in the x -direction of the link frame to which it is attached, and zero otherwise.

6.4 Simulation results

The proposed control scheme was simulated on the vehicle described in Section 2.1.2. Simulations were performed for two cases: one where the links are each passively stabilized like described in Section 2.1.2, and one where each link of the vehicle has uniform density, i.e. the CG coincides with the CB for each link. In both cases, the vehicle is neutrally buoyant. The direction of gravity is set to be directly downwards, parallel to the inertial z -axis.

The reference velocity is $u_d = 0.5$ m/s, and $k_e = \frac{u_d}{10}$. The desired body shape is given by $\alpha = 0.7$. The attitude control gains are chosen to be $k_P = 2$ and $k_D = 19$ for the AIAUV with uniform density, while for the passively stabilized case, $k_D = 12$. Controller parameters were chosen as $k_{\theta,P} = 30$, $k_{\theta,D} = 50$, $k_{1,P} = 100$, $k_{1,I} = 5$. The vehicle starts in a position parallel to the path.

Snapshots from the simulation of the vehicle with uniform density links are shown in Figure 6.3. The cross-track errors from the simulations are shown in Figure 6.4,

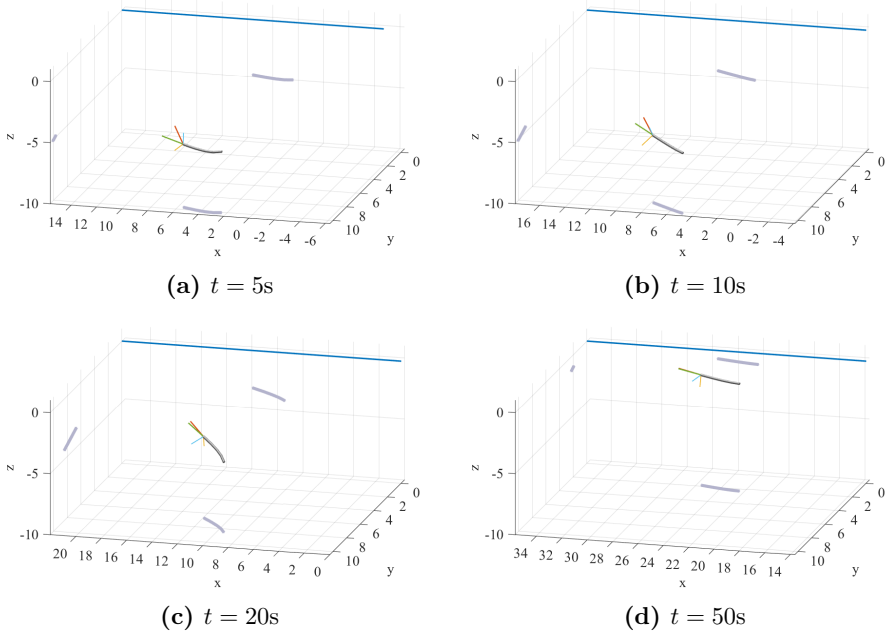


Figure 6.3: Simulation snapshots of the vehicle with uniform density links. Path is shown in blue, current pointing direction in green, reference pointing direction in red, and end-effector frame y - and z -axes in yellow and light blue, respectively.

and surge velocity in Figure 6.5. Figures 6.6, 6.7 and 6.8 show the error angle $\tilde{\phi}$, angular velocity error $\tilde{\omega}^e$ and base angles $\theta_{y,0}$, $\theta_{z,0}$, respectively. In addition, Figure 6.9 shows the inertial-frame representation $\mathbf{x}^I$ of the end-effector x -axis, for the case of uniform density.

As can be seen from Figures 6.4 and 6.5, the control objectives (6.38) are successfully met. Furthermore, Figures 6.6 and 6.7 show that the errors $\tilde{\phi}$ and $\tilde{\omega}^e$ converge to zero, albeit slowly in the passively stabilized case. This is achieved despite the fact that the control law (6.13) is used only indirectly to provide joint angle references.

In the case of following a straight path, the reference pointing direction converges to a constant. As can be seen from Figure 6.9, the reference direction is tracked well, but there is still a small deviation before the reference itself has converged to a constant direction. It is therefore not clear from these results whether the proposed means of direction control for AIAUVs would perform well is the reference varied more over time. The dynamics of the turning motion with joint angles as inputs have not been derived, and have therefore not been cancelled or otherwise compensated by the proposed method of direction control. Deriving these dynamics would also allow to examine what conditions the system must meet to achieve exact tracking using joints for direction control. In addition, due to the choice of a simple joint controller, the joint references cannot be expected to be tracked perfectly either. Since the references are effectively treated as setpoints, a delay in the joint reference tracking is to be expected.

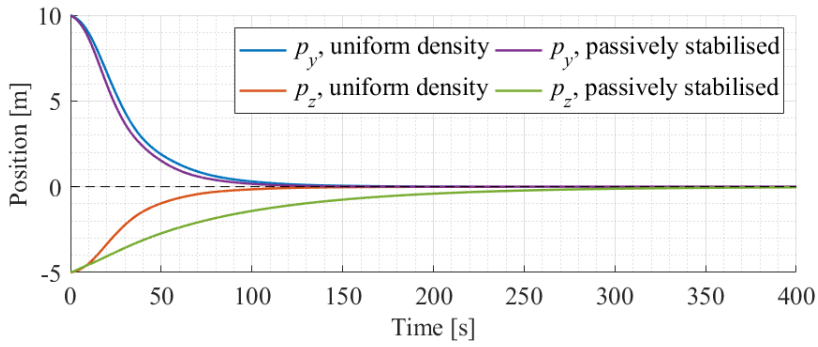


Figure 6.4: Cross-track errors

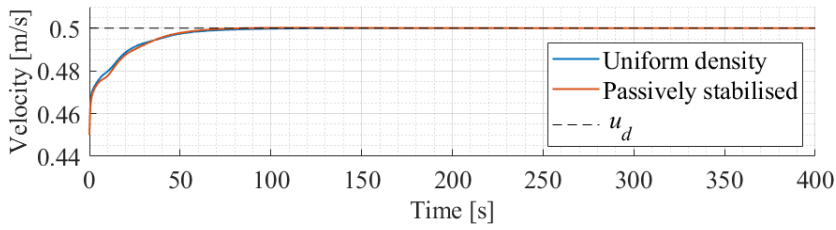


Figure 6.5: Surge velocity

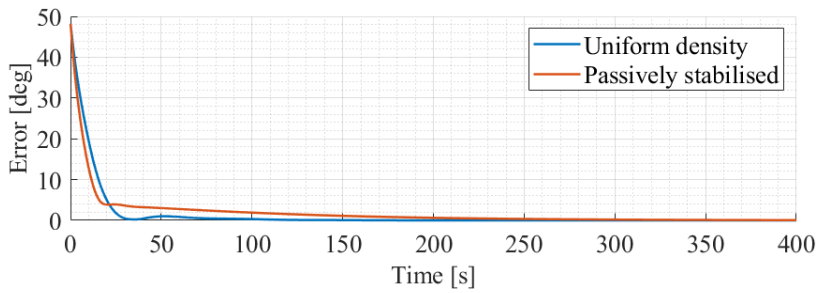


Figure 6.6: Error angle $\tilde{\phi}$ between current and reference pointing direction

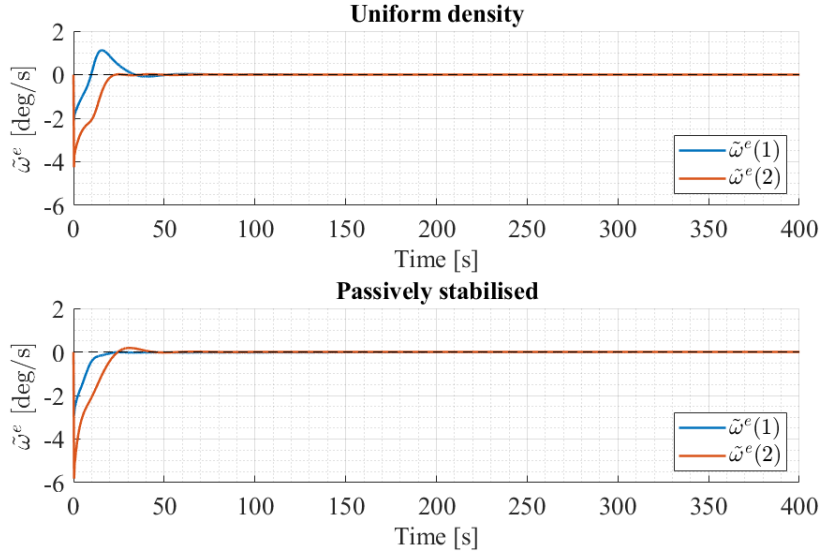
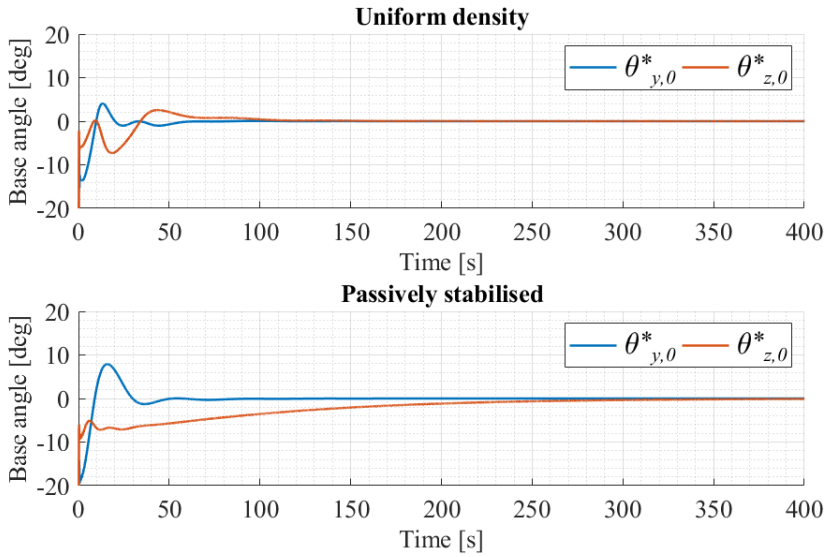
Figure 6.7: Angular velocity error $\tilde{\omega}^e$ 

Figure 6.8: Base angles for the joint references

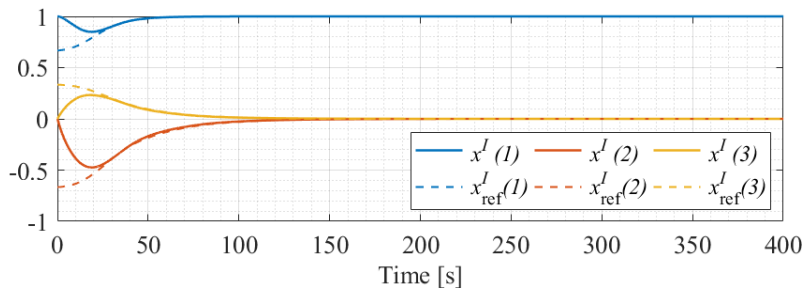


Figure 6.9: End-effector and reference pointing direction in the inertial frame, for the vehicle with uniform density.

As previously mentioned, the proposed control method does not perform as well when the AIAUV is passively stabilized. This can especially be seen in the slower convergence of the cross-track error in the z -direction in Figure 6.4. In this case, not only the joints but also the AIAUV as a whole is influenced by hydrostatic restoring forces which will rotate it, and which are not taken into account in the attitude control method. In contrast, the control law (6.13) cancels all of the dynamics which influence the reduced attitude in order to achieve tracking of the reference. In the simulated scenario, the path is horizontal, i.e. perpendicular to the direction of gravity, so that no hydrostatic restoring forces will act on the AIAUV once it travels along the path. If, instead, the path was sloping, it is likely that the passively stabilized AIAUV would need to stay curved while following the path, thus being unable to keep its head both on the path and pointing along it.

While energy-efficiency has been an important motivation for the development of a path following method for AIAUV which does not rely on use of additional thrusters for turning, the energy consumption of the method remains to be investigated.

6.5 Chapter summary

In this chapter, a control law for reduced attitude tracking was developed for a general type of system with two control torques, and shown to give asymptotic tracking. A method for controlling the pointing direction of an AIAUV by curving its body was then proposed, using the previously developed control law to provide references for the joints of the AIAUV. The method was used to follow a reference guiding the AIAUV onto and along a straight path, and the applicability of the complete path following method was demonstrated by simulations.

Part II

Practical Asymptotic Stability and Control of Underwater Snake Robots

Chapter 7

Uniform Practical Asymptotic Stability

This chapter introduces background material on local uniform practical asymptotic stability, and presents a new theorem for uniform practical asymptotic stability of cascades consisting of uniformly practically asymptotically stable (UPAS) subsystems.

Contributions of this chapter

The contribution of this chapter is a new theorem for uniform practical asymptotic stability of cascades consisting of UPAS subsystems, which relaxes the conditions of the cascade theorem presented in [113]. In this chapter a new intermediate result is proved, namely that solutions starting sufficiently near the origin are bounded as a consequence of the other conditions posed in the cascade theorem from [113], which eliminates the requirement of boundedness of the solution as an explicit condition to apply the theorem.

Publications

This chapter is based on [115].

Organization of this chapter

A brief introduction of the notion of practical asymptotic stability, and some background is given in Section 7.1. Necessary definitions and other precursory theory are given in Section 7.2, before the main result for cascaded systems is stated and proved in Section 7.3. A brief summary of the chapter is given in Section 7.4

7.1 Introduction

In some cases, such as in the presence of disturbances or for systems with an oscillating motion, it may not be possible to make the trajectories of a system converge asymptotically to the origin. Instead, in many situations solutions will converge to and stay within a neighbourhood of the origin, i.e. the solutions are (uniformly)

ultimately bounded. The size of this neighbourhood, determining how close to the origin the solutions will remain, may depend on system parameters, and preferably on control parameters which can be chosen. If the neighbourhood to which the trajectories of the system converge can be made arbitrarily small by appropriately choosing parameters, the system is said to be uniformly practically asymptotically stable (UPAS) [88]. If the trajectories converge to this neighbourhood from any initial condition, the system is uniformly globally practically asymptotically stable (UGPAS) [29]. If the trajectories converge only from initial conditions within a certain area, but this area can be made arbitrarily large by choice of system parameters, the system is said to be uniformly semiglobally practically stable (USPAS) [30].

Theorems to establish uniform global or semiglobal practical asymptotic stability of cascaded systems consisting of two interconnected UGPAS or USPAS subsystems, respectively, were presented in [28–30]. The USPAS property requires that it is possible to make the region of attraction arbitrarily large by choice of parameters. However, in some applications it is not possible to enlarge the region of attraction beyond a certain area, and it is of interest to study local stability properties. To this end, local uniform practical stability was defined in [88]. More general sufficient conditions for UPAS were given in [113], together with a preliminary theorem for UPAS of cascaded systems consisting of two UPAS subsystems. However, to prove that the cascaded system is UPAS, [113] requires solutions starting sufficiently close to the origin to be bounded as an explicit condition. The requirement of bounded solutions will be addressed in this chapter.

7.2 Preliminaries

This section introduces the notion of practical asymptotic stability and presents previous work on the topic. This section is based on material from [30, 88, 113].

7.2.1 Definitions

Stability properties of a ball

The following definitions are from [30]. Consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad (7.1)$$

with $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$ and $\mathbf{f} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous in t , and Lipschitz in $\mathbf{x}$. Additionally, δ and Δ are defined as non-negative numbers such that $\Delta > \delta$. The region inside a closed ball of radius Δ , but outside a ball of radius δ is denoted $H(\delta, \Delta)$, defined as $H(\delta, \Delta) = \{\mathbf{x} \in \mathbb{R}^n \mid \delta \leq \|\mathbf{x}\| \leq \Delta\}$.

Definition 7.1 (Uniform stability of a ball). The ball $\bar{\mathcal{B}}_\delta$ is said to be US on $\mathcal{B}_\Delta$ for system (7.1) if there exists a class $\mathcal{K}_\infty$ function α such that the solutions of (7.1) from any initial state $\mathbf{x}_0 \in \bar{\mathcal{B}}_\Delta$ and any initial time $t_0 \in \mathbb{R}_{\geq 0}$ satisfy $\|\mathbf{x}(t, t_0, \mathbf{x}_0)\|_\delta \leq \alpha(\|\mathbf{x}_0\|)$ for all $t \geq t_0$.

Definition 7.2 (Uniform attractiveness of a ball). The ball $\bar{\mathcal{B}}_\delta$ is said to be UA on $\mathcal{B}_\Delta$ for system (7.1) if there exists a class $\mathcal{L}$ function σ such that the solutions of (7.1)

from any initial state $\mathbf{x}_0 \in \bar{\mathcal{B}}_\Delta$ and initial time $t_0 \in \mathbb{R}_{\geq 0}$ satisfy $\|\mathbf{x}(t, t_0, \mathbf{x}_0)\|_\delta \leq \sigma(t - t_0)$ for all $t \geq t_0$.

Definition 7.3 (Uniform asymptotic stability of a ball). The ball $\bar{\mathcal{B}}_\delta$ is said to be UAS on $\bar{\mathcal{B}}_\Delta$ for the system (7.1) if it is both US and UA on $\bar{\mathcal{B}}_\Delta$.

Uniform practical asymptotic stability

Consider a time-varying dynamical system on the following form:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\vartheta}), \quad (7.2)$$

with $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$, and where $\boldsymbol{\vartheta} \in \mathbb{R}^m$ is a constant parameter, typically a control gain that can be tuned. The dynamics $\mathbf{f} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are locally Lipschitz in $\mathbf{x}$ and piece-wise continuous in t and $\boldsymbol{\vartheta}$.

Definition 7.4 (Uniform practical asymptotic stability [88]). Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (7.2) is said to be UPAS on Θ if for some $\Delta > 0$, for any positive $\delta < \Delta$ there exists $\boldsymbol{\vartheta}^*(\delta) \in \Theta$ such that the ball $\bar{\mathcal{B}}_\delta$ is UAS on $\bar{\mathcal{B}}_\Delta$ for the system (7.2).

Note that this is a local adaptation of UGPAS and USPAS as presented in [29, 30]. Specifically, if the radius Δ of $\bar{\mathcal{B}}_\Delta$ can be increased arbitrarily by the choice of parameters, the definition extends to that of USPAS, and if $\Delta \rightarrow \infty$ regardless of the parameters selected, then it becomes the definition of UGPAS.

The set of all possible parameters which render $\bar{\mathcal{B}}_\delta$ UAS on $\bar{\mathcal{B}}_\Delta$ for the system (7.2) is UPAS is defined as

$$\mathcal{D}_f(\delta, \Delta) = \{\boldsymbol{\vartheta} \in \mathbb{R}^m \mid \bar{\mathcal{B}}_\delta \text{ is UAS on } \bar{\mathcal{B}}_\Delta \text{ for (7.2)}\}. \quad (7.3)$$

7.2.2 Lyapunov Sufficient Conditions for UPAS

The following theorem from [113] is an adaptation of [30, Theorem 10] and gives sufficient conditions for a system on the form (7.2) to be UPAS.

Theorem 7.1 (Lyapunov sufficient conditions for UPAS). *Suppose that there exists a $\Delta > 0$ such that, for any positive $\delta < \Delta$ there exist a parameter $\boldsymbol{\vartheta}^*(\delta) \in \Theta$, a continuously differentiable function $V_\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, and class $\mathcal{K}$ -functions $\underline{\alpha}_\delta$, $\bar{\alpha}_\delta$ and α_δ , such that for all $\mathbf{x} \in H(\delta, \Delta)$, and all $t \in \mathbb{R}_{\geq 0}$*

$$\underline{\alpha}_\delta(\|\mathbf{x}\|) \leq V_\delta(t, \mathbf{x}) \leq \bar{\alpha}_\delta(\|\mathbf{x}\|), \quad (7.4)$$

$$\frac{\partial V_\delta}{\partial t}(t, \mathbf{x}) + \frac{\partial V_\delta}{\partial \mathbf{x}}(t, \mathbf{x}) \mathbf{f}(t, \mathbf{x}, \boldsymbol{\vartheta}^*) \leq -\alpha_\delta(\|\mathbf{x}\|), \quad (7.5)$$

$$\lim_{\delta \rightarrow 0} \alpha_\delta^{-1} \circ \bar{\alpha}_\delta(\delta) = 0. \quad (7.6)$$

then the system (7.2) is UPAS on the parameter set Θ .

It is worth stressing that the $\mathcal{K}$ functions in (7.4), (7.5) typically depend on the value of the parameter $\boldsymbol{\vartheta}$, which is itself tuned to reach a given precision δ . In other words, these $\mathcal{K}$ functions typically depend on δ , as indicated by the subscript. This is what makes condition (7.6) non-trivial.

The conditions stated in Theorem 7.1 are similar to those given in [88, Corollary 2], but differ in how the bounds (7.4) which V_δ has to satisfy are defined. In [88, Corollary 2] the bounds are given in terms of powers of the norm of the state, in particular the same power for both the lower and upper bound, and the bounds also depend on a set of functions of the system parameters. In Theorem 7.1 the bounds are given in terms of arbitrary class $\mathcal{K}$ functions which depend on δ rather than the parameter $\boldsymbol{\vartheta}$.

7.3 UPAS of cascades

This section presents the (main) contributions of this chapter, which are results on the stability of cascaded systems consisting of UPAS subsystems. A preliminary version of these results was presented in [113], but with stricter conditions.

7.3.1 Main results

Consider the cascaded system

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(t, \mathbf{x}_1, \boldsymbol{\vartheta}_1) + \mathbf{g}(t, \mathbf{x}, \boldsymbol{\vartheta})\mathbf{x}_2, \quad (7.7a)$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(t, \mathbf{x}_2, \boldsymbol{\vartheta}_2), \quad (7.7b)$$

where $\mathbf{x} = [\mathbf{x}_1^\top, \mathbf{x}_2^\top]^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ are the states, $\boldsymbol{\vartheta} = [\boldsymbol{\vartheta}_1^\top, \boldsymbol{\vartheta}_2^\top]^\top \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ are the parameters, $t \in \mathbb{R}_{\geq 0}$, $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{g}$ are locally Lipschitz in the states and parameters, and piece-wise continuous in time. The subsystems (7.7a) and (7.7b) are the driven and driving subsystem, respectively, and

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(t, \mathbf{x}_1, \boldsymbol{\vartheta}_1) \quad (7.8)$$

will be referred to as the nominal dynamics of the driven subsystem. Additionally, we make the assumption

Assumption 7.1 (Boundedness of the interconnection term). The function $\mathbf{g}$ is uniformly bounded both in time and parameters, i.e. there exists a nondecreasing function $G : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that, for all $\mathbf{x} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, all $\boldsymbol{\vartheta} \in \Theta_1 \times \Theta_2$ and all $t \in \mathbb{R}_{\geq 0}$

$$\|\mathbf{g}(t, \mathbf{x}, \boldsymbol{\vartheta})\| \leq G(\|\mathbf{x}\|). \quad (7.9)$$

To establish UPAS of cascaded systems on the form given by (7.7), [30, Theorem 13] is adapted for the case of local stability properties in [113], and the conditions are further relaxed here. The following assumptions are made:

Assumption 7.2 (UPAS of the driving subsystem). The driving system (7.7b) is UPAS on Θ_2 .

Assumption 7.3 (UPAS of the driven subsystem). For some $\Delta_1 > 0$ and any δ_1 such that $\Delta_1 > \delta_1 > 0$, there exist a parameter $\boldsymbol{\vartheta}_1^*(\delta_1) \in \Theta_1$, a continuously differentiable function V_{δ_1} , and class $\mathcal{K}$ functions $\underline{\alpha}_{\delta_1}$, $\bar{\alpha}_{\delta_1}$ and α_{δ_1} such that the conditions (7.4), (7.5) and (7.6) are satisfied with these functions for the nominal system (7.8), and a continuous positive non-decreasing function c_{δ_1} exists such that for all $\mathbf{x}_1 \in H(\delta_1, \Delta_1)$ and all $t \in \mathbb{R}_{\geq 0}$

$$\left\| \frac{\partial V_{\delta_1}}{\partial \mathbf{x}_1}(t, \mathbf{x}_1) \right\| \leq c_{\delta_1}(\|\mathbf{x}_1\|). \quad (7.10)$$

In [113], it was also assumed that solutions of (7.7) starting sufficiently close to the origin must be bounded. However, it can be shown that boundedness of solutions follows from Assumptions 7.1-7.3. The following proposition is therefore made here:

Proposition 7.1 (Boundedness of solutions). *Given Assumptions 7.1-7.3, there exist positive constants Δ_0 , δ_1 , δ_2 , γ such that the trajectories of the cascaded system (7.7) with $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^*$, where $\boldsymbol{\vartheta}_1^*(\delta_1) \in \Theta_1$ is given by Assumption 7.3 and $\boldsymbol{\vartheta}_2^* \in D_{f_2}(\delta_2, \Delta_2) \cap \Theta_2$, satisfy*

$$\|\mathbf{x}_0\| \leq \gamma \Rightarrow \|\mathbf{x}(t, t_0, \mathbf{x}_0, \boldsymbol{\vartheta}^*)\| < \Delta_0, \forall t \geq t_0 \quad (7.11)$$

The proof of Proposition 7.1 is given in Section 7.3.2. It is worth noting that the proof only requires $\|\mathbf{x}_2(t)\|_{\delta_2}$ to be bounded by a class $\mathcal{K}$ function, not class $\mathcal{KL}$. As a consequence, the conditions of Proposition 7.1 could be relaxed for the driving subsystem, requiring only that there is a ball $\mathcal{B}_{\delta_2}$ which is US on some $\mathcal{B}_{\Delta_2}$. With Proposition 7.1, a new theorem which relaxes the conditions of [113, Theorem 1] is stated here:

Theorem 7.2. *Under Assumptions 7.1-7.3 the cascaded system (7.7) is UPAS on $\Theta_1 \times \Theta_2$.*

The proof of Theorem 7.2 is given in Section 7.3.3.

7.3.2 Proof of Proposition 7.1

The main part of this proof consists of showing that solutions of (7.7) starting sufficiently close to the origin do not exit the ball $\mathcal{B}_{\Delta_0}$. Then, since by assumption the right-hand side of (7.7) is locally Lipschitz everywhere in the state space, by [78, Theorem 3.3] solutions starting at some time t_0 and which lie entirely in the compact set $\mathcal{B}_{\Delta_0}$ are defined for all time $t \geq t_0$, completing the statement (7.11). The line of reasoning in this proof is inspired by the proof of [67, Theorem 5].

We will prove that solutions of (7.7) starting sufficiently close to the origin do not exit $\mathcal{B}_{\Delta_0}$ by contradiction. Let Assumptions 7.1-7.3 be satisfied, and assume that for any Δ_0 , any δ_1 , any δ_2 and any $\gamma > 0$, there exist solutions of (7.7) with $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^*$, generated by δ_1 , δ_2 , starting from initial conditions $\mathbf{x}_0 \in \mathcal{B}_\gamma$ and which cross the boundary of $\mathcal{B}_{\Delta_0}$. The parameter $\boldsymbol{\vartheta}^*$ consists of $\boldsymbol{\vartheta}_1^* = \boldsymbol{\vartheta}_1^*(\delta_1)$ generated by Assumption 7.3 and $\boldsymbol{\vartheta}_2^* \in D_{f_2}(\delta_2, \Delta_2) \cap \Theta_2$. In other words, for any Δ_0 , any δ_1 and any δ_2 , there are solutions of (7.7), with $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^*$, starting arbitrarily close to the

origin which reach the boundary of $\mathcal{B}_{\Delta_0}$. Then we can construct a sequence of initial conditions $(\mathbf{x}_n^0)_{n \in \mathbb{N}} \subset \mathcal{B}_{\Delta_0}$, $\lim_{n \rightarrow \infty} \mathbf{x}_n^0 = 0$ where for each n , there exists an initial time $t_n^0 \geq 0$ and a finite $T_n > 0$ such that the solution of (7.7) starting in $\mathbf{x}_n^0$ at time t_n^0 reaches the boundary of $\mathcal{B}_{\Delta_0}$ after a time interval T_n , i.e.

$$\begin{aligned} t_n^0 \leq t < t_n^0 + T_n &\Rightarrow \|\mathbf{x}(t, t_n^0, \mathbf{x}_n^0, \boldsymbol{\vartheta}^*)\| < \Delta_0, \\ \|\mathbf{x}(t_n^0 + T_n, t_n^0, \mathbf{x}_n^0, \boldsymbol{\vartheta}^*)\| &= \Delta_0 \end{aligned} \quad (7.12)$$

Since we assume (7.12) has to hold for any Δ_0 , let $\Delta_0 < \min(\Delta_1, \Delta_2)$, where Δ_2 and Δ_1 follow from Assumptions 7.2 and 7.3, respectively.

Now, we make use of the system properties similarly to the reasoning in the stability proof in [30]. Consider a function V_{δ_1} generated by Assumption 7.3, and use [30, Lemma 27] with $X = H(\delta_1, \Delta_1)$ and with the choice $k = 1$ to generate a function $\mathcal{V}_{\delta_1}$, class $\mathcal{K}$ functions $\underline{\chi}_{\delta_1}$, $\bar{\chi}_{\delta_1}$, and a continuous non-decreasing function d_{δ_1} such that $\forall \mathbf{x} \in X$ and $\forall t \in \mathbb{R}_{\geq 0}$

$$\underline{\chi}_{\delta_1}(\|\mathbf{x}\|) \leq \mathcal{V}_{\delta_1}(t, \mathbf{x}) \leq \bar{\chi}_{\delta_1}(\|\mathbf{x}\|) \quad (7.13a)$$

$$\frac{\partial \mathcal{V}_{\delta_1}}{\partial t}(t, \mathbf{x}) + \frac{\partial \mathcal{V}_{\delta_1}}{\partial \mathbf{x}}(t, \mathbf{x})f(t, \mathbf{x}) \leq -\mathcal{V}_{\delta_1}(t, \mathbf{x}) \quad (7.13b)$$

$$\left\| \frac{\partial \mathcal{V}_{\delta_1}}{\partial \mathbf{x}}(t, \mathbf{x}) \right\| \leq d_{\delta_1}(\|\mathbf{x}\|) \quad (7.13c)$$

hold for $\mathcal{V}_{\delta_1}$, for some $\delta_1 < \Delta_0$. From [30, Lemma 27], we have that for any $s \in \mathbb{R}_{\geq 0}$

$$\underline{\chi}_{\delta_1}^{-1} \circ \bar{\chi}_{\delta_1}(s) = \underline{\alpha}_{\delta_1}^{-1} \circ \bar{\alpha}_{\delta_1}(s). \quad (7.14)$$

The Lie derivative of $\mathcal{V}_{\delta_1}$ with $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^*$ yields

$$\dot{\mathcal{V}}_{\delta_1} = \frac{\partial \mathcal{V}_{\delta_1}}{\partial t} + \frac{\partial \mathcal{V}_{\delta_1}}{\partial \mathbf{x}_1}(f_1(t, \mathbf{x}_1, \boldsymbol{\vartheta}_1^*) + g(t, \mathbf{x}, \boldsymbol{\vartheta}^*)\mathbf{x}_2). \quad (7.15)$$

Then $\forall \mathbf{x}_1 \in H(\delta_1, \Delta_1)$ and $\forall t \geq 0$

$$\dot{\mathcal{V}}_{\delta_1} \leq -\mathcal{V}_{\delta_1} + \left\| \frac{\partial \mathcal{V}_{\delta_1}}{\partial \mathbf{x}_1} \right\| \|g(t, \mathbf{x}, \boldsymbol{\vartheta}^*)\| \|\mathbf{x}_2\|, \quad (7.16a)$$

$$\leq -\mathcal{V}_{\delta_1} + d_{\delta_1}(\|\mathbf{x}_1\|)G(\|\mathbf{x}\|)\|\mathbf{x}_2\|. \quad (7.16b)$$

Let $\mathbf{x}_i(t, \mathbf{x}_n^0) = \mathbf{x}_i(t, t_n^0, \mathbf{x}_n^0, \boldsymbol{\vartheta}^*)$ for $i = 1, 2$, and $v_{\delta_1, \Delta_1}(t, \mathbf{x}_n^0) = \mathcal{V}_{\delta_1, \Delta_1}(t, \mathbf{x}_1(t, \mathbf{x}_n^0))$. Since $\mathbf{x}_n^0 \in \bar{\mathcal{B}}_{\Delta_0} \subset \bar{\mathcal{B}}_{\Delta_2}$, by Assumption 7.2 there exists a class $\mathcal{KL}$ function β_2 s.t.

$$\|\mathbf{x}_2(t, \mathbf{x}_n^0)\| \leq \beta_2(\|\mathbf{x}_2(t_n^0, \mathbf{x}_n^0)\|, t - t_n^0) + \delta_2 \quad \forall t \geq t_n^0 \quad (7.17)$$

holds for any $\delta_2 < \Delta_2$. Now define

$$\Gamma_n = \{t \in [t_n^0, t_n^0 + T_n] \mid \delta_1 \leq \|\mathbf{x}_1(t, \mathbf{x}_n^0)\| \leq \Delta_1\}. \quad (7.18)$$

Note that by (7.12) and since $\delta_1 < \Delta_0 \leq \Delta_1$, the set Γ_n is non-empty (since it contains at least $t_n^0 + T_n$, and by continuity of solutions some interval preceding that).

Combining (7.16), (7.12) and (7.17), we have that $\forall t \in \Gamma_n$,

$$\begin{aligned} \dot{v}_{\delta_1}(t, \mathbf{x}_n^0) &\leq -v_{\delta_1}(t) + d_{\delta_1}(\Delta_1)G(\Delta_0) (\beta_2(\|\mathbf{x}_2(t_n^0, \mathbf{x}_n^0)\|, t - t_n^0) + \delta_2) \\ &\leq -v_{\delta_1}(t) + c_3(\|\mathbf{x}_n^0\|) \end{aligned} \quad (7.19)$$

where

$$c_3(\cdot) = d_{\delta_1}(\Delta_1)G(\Delta_1) (\beta_2(\cdot, 0) + \delta_2). \quad (7.20)$$

Integrating (7.19) over Γ_n in the same manner as is done in the proof of Lemma 28 in [30], and writing $\mathbf{x}(t_n^0 + T_n, \mathbf{x}_n^0)$ for $\mathbf{x}(t_n^0 + T_n, t_n^0, \mathbf{x}_n^0, \boldsymbol{\vartheta}^*)$, we have

$$\begin{aligned} \|\mathbf{x}(t_n^0 + T_n, \mathbf{x}_n^0)\| &\leq \delta_1 + \underline{\alpha}_{\delta_1}^{-1} (\bar{\alpha}_{\delta_1}(\delta_1) + c_3(\|\mathbf{x}_n^0\|)) \\ &\quad + \underline{\alpha}_{\delta_1}^{-1} (\bar{\alpha}_{\delta_1}(\|\mathbf{x}_n^0\|)e^{-T_n} + c_3(\|\mathbf{x}_n^0\|)) \\ &\leq \delta_1 + \underline{\alpha}_{\delta_1}^{-1} (\bar{\alpha}_{\delta_1}(\delta_1) + c_3(\|\mathbf{x}_n^0\|)) \\ &\quad + \underline{\alpha}_{\delta_1}^{-1} (\bar{\alpha}_{\delta_1}(\|\mathbf{x}_n^0\|) + c_3(\|\mathbf{x}_n^0\|)). \end{aligned} \quad (7.21)$$

Define

$$\delta_3 = \delta_1 + \underline{\alpha}_{\delta_1}^{-1} (\bar{\alpha}_{\delta_1}(\delta_1) + c_3(0)) + \underline{\alpha}_{\delta_1}^{-1} (c_3(0)), \quad (7.22)$$

$$\begin{aligned} \eta(s) &= \underline{\alpha}_{\delta_1}^{-1} (\bar{\alpha}_{\delta_1}(\delta_1) + c_3(s)) + \underline{\alpha}_{\delta_1}^{-1} (\bar{\alpha}_{\delta_1}(s) + c_3(s)) \\ &\quad - \underline{\alpha}_{\delta_1}^{-1} (\bar{\alpha}_{\delta_1}(\delta_1) + c_3(0)) - \underline{\alpha}_{\delta_1}^{-1} (c_3(0)), \end{aligned} \quad (7.23)$$

where, since c_3 is a continuous non-increasing function, η is a class $\mathcal{K}$ function. By Assumptions 7.2 and 7.3, we can choose δ_1, δ_2 respectively, to generate corresponding parameters $\boldsymbol{\vartheta}_1^*, \boldsymbol{\vartheta}_2^*$ such that $\delta_3 < \Delta_0 - 2\varepsilon$ for some $\varepsilon > 0$. Furthermore, since η is a class $\mathcal{K}$ function and $\lim_{n \rightarrow \infty} \mathbf{x}_n^0 = 0$, we can find an N such that $\eta(\|\mathbf{x}_n^0\|) < \varepsilon \forall n \geq N$. This gives

$$\|\mathbf{x}(t_n^0 + T_n, t_n^0, \mathbf{x}_n^0, \boldsymbol{\vartheta}^*)\| \leq \Delta_0 - \varepsilon \neq \Delta_0 \quad \forall n \geq N \quad (7.24)$$

which contradicts (7.12). Taking $\gamma < \eta^{-1}(\varepsilon)$, for a $\Delta_0 < \min(\Delta_1, \Delta_2)$, we have thus found a δ_1, δ_2 and γ such that no solution starting from $\mathbf{x}_0 \in \mathcal{B}_\gamma$ can exit $\mathcal{B}_{\Delta_0}$. $\square$

7.3.3 Proof of Theorem 7.2

The proof of Theorem 7.2 follows from the proof of Theorem 13 in [30] by setting Δ_1 to be constant.

Consider $\mathcal{V}_{\delta_1}$ as given by (7.13). For any δ_1 that satisfies $\Delta_1 > \delta_1 > 0$ it follows from (7.6) and (7.14) that

$$\lim_{\delta_1 \rightarrow 0} \underline{\chi}_{\delta_1}^{-1} \circ \bar{\chi}_{\delta_1}(\delta_1) = 0. \quad (7.25)$$

Define $\Delta = \min\{\Delta_1, \Delta_2, \gamma\}$ where γ is given by Proposition 7.1. Choose any $\boldsymbol{\vartheta}_1^* \in \Theta_1$ satisfying Assumption 7.3 and $\boldsymbol{\vartheta}_2^* \in \mathcal{D}f_2(\delta_2, \Delta_2) \cap \Theta_2$ such that Proposition 7.1 holds. We first show that there exists a positive δ_3 such that the ball $\bar{\mathcal{B}}_{\delta_3}$ is uniformly stable by constructing $\eta \in \mathcal{K}$ such that $\forall \mathbf{x}_0 \in \bar{\mathcal{B}}_\Delta$

$$\|\mathbf{x}_1(t, t_0, \mathbf{x}_0, \delta)\|_{\delta_3} \leq \eta(\|\mathbf{x}_0\|). \quad (7.26)$$

We use this property to prove that a ball larger than $\bar{\mathcal{B}}_{\delta_3}$ is UA on $\bar{\mathcal{B}}_{\Delta}$, and we construct a $\mathcal{KL}$ estimate for the solutions. Finally, we show that the estimates of the ball to which solutions converge can be arbitrarily diminished. The time derivative of $\mathcal{V}_{\delta_1}$ with $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^*$ is given by (7.15).

Additionally, we define the subset

$$\Gamma = \{t \geq t_0 \mid \delta_1 \leq \|\mathbf{x}_1(t, t_0, \mathbf{x}_0, \boldsymbol{\vartheta}^*)\| \leq \Delta_1\}. \quad (7.27)$$

We now use $\mathbf{x}_1(t)$ for $\mathbf{x}_1(t, t_0, \mathbf{x}_0, \boldsymbol{\vartheta}^*)$ and $v_{\delta_1}(t)$ for $\mathcal{V}_{\delta_1}(t, \mathbf{x}_1(t))$. We then have $\forall \mathbf{x}_0 \in \bar{\mathcal{B}}_{\Delta}, \forall t \in \Gamma$,

$$\dot{v}_{\delta_1}(t) \leq -v_{\delta_1}(t) + d_{\delta_1}(\Delta_1)G(\Delta_1)\|\mathbf{x}_2(t)\|. \quad (7.28)$$

Like previously, since by Assumption 7.2 the driving system (7.7b) is UPAS, there is a class $\mathcal{KL}$ function

$$\|\mathbf{x}_2(t)\| \leq \beta_2(\|\mathbf{x}_{20}\|, t - t_0) + \delta_2, \forall t \geq 0. \quad (7.29)$$

for $\mathbf{x}_{20} \in \bar{\mathcal{B}}_{\Delta_2}$. It follows that $\forall \mathbf{x}_0 \in \bar{\mathcal{B}}_{\Delta}$ and $\forall t \in \Gamma$

$$\dot{v}_{\delta_1}(t) \leq -v_{\delta_1}(t) + d_{\delta_1}(\Delta_1)G(\Delta_1)(\beta_2(\|\mathbf{x}_{20}\|, t - t_0) + \delta_2), \quad (7.30)$$

which implies that

$$\dot{v}_{\delta_1}(t) \leq -v_{\delta_1}(t) + c_3(\|\mathbf{x}_0\|), \forall \mathbf{x}(t) \in H(\delta_1, \Delta_1), \quad (7.31)$$

where c_3 is the same function as defined by (7.20).

We now consider [30, Lemma 28], which allows us to construct a bound with $V = \mathcal{V}_{\delta_1}$, $k = 1$, $c = c_3(\|\mathbf{x}_0\|)$ and $X = H(\delta, \Delta)$. In view of Proposition 7.1, we have that $\forall \mathbf{x}_0 \in \bar{\mathcal{B}}_{\Delta}$ and $\forall t_0 \in \mathbb{R}_{\geq 0}$

$$\|\mathbf{x}(t)\| \leq \delta_1 + \underline{\alpha}_{\delta_1}^{-1}(\bar{\alpha}_{\delta_1}(\delta_1) + c_3(\|\mathbf{x}_0\|)) + \underline{\alpha}_{\delta_1}^{-1}(\bar{\alpha}_{\delta_1}(\|\mathbf{x}_0\|) + c_3(\|\mathbf{x}_0\|)). \quad (7.32)$$

Defining δ_3 and a function η as given by (7.22) and (7.23), respectively, we conclude that, for any $\mathbf{x}_0 \in \bar{\mathcal{B}}_{\Delta}$ and all $t_0 \in \mathbb{R}_{\geq 0}$ it holds that

$$\|\mathbf{x}_1(t)\|_{\delta_3} \leq \eta(\|\mathbf{x}_0\|), \forall t \geq t_0. \quad (7.33)$$

Uniform stability of $\bar{\mathcal{B}}_{\delta_3}$ on $\bar{\mathcal{B}}_{\Delta}$ follows by noticing that η is a class $\mathcal{K}$ function.

To show uniform attractivity of a ball, we again consider (7.30). Because β_2 is a class $\mathcal{KL}$ function, there is a time $t_1 \geq 0$ such that

$$\beta_2(\Delta, t - t_0) \leq \delta_2, \forall t \geq t_0 + t_1. \quad (7.34)$$

Therefore it is implied that

$$\dot{v}_{\delta_1} \leq -v_{\delta_1}(t) + 2d_{\delta_1}(\Delta_1)G(\Delta_1)\delta_2, \forall t \geq t_0 + t_1. \quad (7.35)$$

Applying [30, Lemma 28] again, it follows that for any $\mathbf{x}_0 \in \bar{\mathcal{B}}_{\Delta}$, all $t_0 \in \mathbb{R}_{\geq 0}$ and all $t \geq t_0 + t_1$

$$\begin{aligned} \|\mathbf{x}(t)\|_{\delta_1} &\leq \underline{\alpha}_{\delta_1}^{-1}(\bar{\alpha}_{\delta_1}(\delta_1) + 2d_{\delta_1}(\Delta_1)G(\Delta_1)\delta_2) \\ &\quad + \underline{\alpha}_{\delta_1}^{-1}(\bar{\alpha}_{\delta_1}(\Delta_1)e^{-(t-t_0-t_1)} + 2d_{\delta_1}(\Delta_1)G(\Delta_1)\delta_2). \end{aligned} \quad (7.36)$$

We define

$$t_2 = t_1 + \ln \left(\frac{\bar{\alpha}_{\delta_1}(\Delta_1)}{\bar{\alpha}_{\delta_1}(\delta_1)} \right). \quad (7.37)$$

By inserting (7.37) and defining

$$\delta_4 = \delta_1 + 2\bar{\alpha}_{\delta_1}^{-1}(\bar{\alpha}_{\delta_1}(\delta_1) + 2d_{\delta_1}(\Delta_1)G(\Delta_1)\delta_2), \quad (7.38)$$

it follows that

$$\|\mathbf{x}_1(t, t_0, \mathbf{x}_{10})\|_{\delta_4} = 0, \quad \forall t \geq t_0 + t_2, \quad (7.39)$$

meaning that the set is uniformly attractive. Furthermore, it can be seen that $\delta_3 \leq \delta_4$ from equations (7.22) and (7.38). Let

$$\delta = \max\{\delta_2, \delta_4\}, \quad (7.40)$$

Then we see that from (7.33) it is implied that $\|\mathbf{x}_1(t)\|_{\delta} \leq \eta(\|\mathbf{x}_0\|)$, $\forall t \geq t_0$. Furthermore, by considering (7.39) the following bound can be found $\forall t \geq t_0, \forall \mathbf{x}_0 \in \bar{\mathcal{B}}_{\Delta}$

$$\|\mathbf{x}_1(t)\| \leq \delta + \eta(\|\mathbf{x}_0\|)e^{-(t-t_0-t_2)}. \quad (7.41)$$

Recalling that t_2 is not dependent on t_0 or $\mathbf{x}_0$, and defining

$$\beta(s, t) = \max(\eta(s)e^{-(t-t_2)}, \beta_2(s, t)), \quad (7.42)$$

we can conclude that for any $\mathbf{x}_0 \in \bar{\mathcal{B}}_{\Delta}$

$$\|\mathbf{x}(t)\| \leq \delta + \beta(\|\mathbf{x}_0\|, t - t_0), \quad (7.43)$$

because this is a class $\mathcal{KL}$ function, we conclude that $\bar{\mathcal{B}}_{\delta}$ is UAS on $\bar{\mathcal{B}}_{\Delta}$. To show that δ can be arbitrarily diminished, consider (7.6) in Assumption 7.3, while recalling that d_{δ_1} , G and $\underline{\chi}_{\delta_1}$ are independent of δ_2 . Then δ_4 and by extension δ_3 can be made arbitrarily diminished by choice of δ_1 and δ_2 . Hence it is sufficient to select the parameters $\boldsymbol{\vartheta}_1^* \in \Theta_1$ and $\boldsymbol{\vartheta}_2^* \in \Theta_2$ generated by the chosen δ_1 and δ_2 to conclude that for any δ satisfying $\Delta > \delta > 0$, there exists some parameters such that $\bar{\mathcal{B}}_{\delta}$ is UAS on $\bar{\mathcal{B}}_{\Delta}$. $\square$

7.4 Chapter summary

In this chapter, background material on local uniform practical asymptotic stability was introduced. Then, a new proposition regarding the boundedness of solutions of cascaded systems consisting of UPAS subsystems was presented. Using this proposition, a new theorem for UPAS of such cascaded systems was presented, relaxing the conditions posed in previous work.

Chapter 8

Automatic Alignment of Underwater Snake Robots Operating in Wakes of Bluff Bodies

This chapter investigates the automatic alignment of a USR with the wake forming behind a bluff body due to an unknown, irrotational ocean current. A cascaded control law which aligns the USR with the wake is presented, and the control law enables the USR to automatically realign itself also in the event of slowly changing current direction. It is shown that the controller renders the distance between the USR and the center-line of the vortex street, as well as the distance from the bluff body, UPAS, also in the presence time-varying currents as in a vortex street. The proposed control method is validated through high-fidelity simulations of several scenarios.

Contributions of this chapter

The contributions of this chapter are summarized as follows. A new, simplified model is proposed for the velocity and orientation of a USR following a sinusoidal gait in the presence of time-varying currents. Based on this model, controllers for the velocity and heading of the USR which track desired time-varying references are developed. The resulting closed-loop system, consisting of the velocity and heading controllers combined with a joint controller from [93], is shown to be UPAS.

A new control method for automatic alignment of swimming USRs with a wake behind a bluff body due to an unknown current is proposed by combining the dynamic controllers for joints, velocity and heading with references designed based on WOPC [48], and the resulting closed-loop system is analysed. It is shown that the control method renders the distance between the USR and the centerline of the vortex street, as well as the distance from the bluff body, UPAS in the presence of a time-varying current. The analysis in this chapter holds also for vehicles other than USRs, provided that they are equipped with lower-level controllers capable of tracking heading and forward velocity, providing a stability proof for weather optimal

heading control (WOHC), which is a part of WOPC, for underactuated vehicles. In previous works on WOPC, the ships considered are either fully actuated [48], or stability analysis is not performed [82].

Unlike many previous works on control of swimming USRs, this chapter considers the presence of time-varying currents. Compared to [113], where time-varying currents were also included, the dynamic controllers in this chapter are developed based on more realistic models of the USR dynamics, and no knowledge or measurements of the current are necessary for the control method presented in this chapter.

The proposed control method is validated through high-fidelity simulations that capture the intricate interaction between the USR and the surrounding fluid.

Publications

This chapter is based on [116].

Organization of this chapter

This chapter is organized as follows. First, an introduction is given in Section 8.1. The model of the USR and the control of joint angles, heading, and tangential velocity are presented in Section 8.2. In Section 8.3, the heading and tangential velocity references are designed, and it is proven that the proposed control system practically asymptotically stabilizes the orientation and position of the USR to achieve alignment with the current at a desired distance from a bluff body. The high-fidelity simulator used to validate the control design is introduced in Section 8.4, and the simulation setup and results are presented and discussed in Section 8.5. Finally, a short summary of the chapter is provided in Section 8.6.

8.1 Introduction

Power delivery remains a challenge for AUVs. Battery constraints limit their operational time, while tethers would limit their operational area and autonomy. Improving the energy efficiency of these systems would be a significant step forward in our attempts to design efficient AUVs. To overcome these challenges and achieve truly autonomous operation, pursuing energy autonomy is an important next step.

Previous studies have suggested the vortex wakes which form behind bluff bodies as a promising avenue for energy harvesting in snake-like structures. In [1], piezoelectric membranes were placed in the wakes of bluff bodies, and these thin, snake-like membranes exhibited lock-in behaviour with the vortex streets, a prerequisite for achieving an optimal coupling where the structure has a minimal damping effect on the wake itself. The energy harvesting capabilities of a three-link articulated swimmer were investigated numerically through simulations in [13]. Studies also suggest that the ability to harvest energy can improve the swimming efficiency of USRs [12, 155]. In [12], the high-fidelity simulation method from [13] was used to demonstrate that a linked swimmer tracking the wake of a bluff body saves energy when compared to swimming in a free flow. The controller developed in [12] achieves tracking of the wake in the simulation, but the stability of the control system is not analysed. An

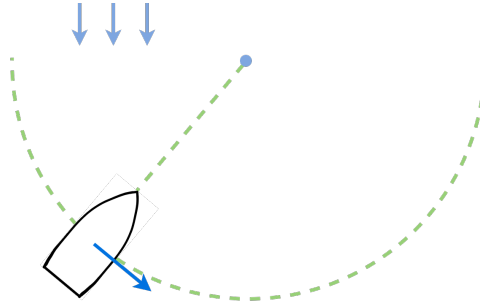


Figure 8.1: Illustration of the concept of WOPC.

extremum-seeking control scheme was used by [114] to find the optimal positioning of a USR for energy harvesting in a vortex street, and high-fidelity simulation results show that the USR converges to a vicinity of the optimal position. The approach in [114] assumes that the USR is already in the vortex street. In order for the USR to transition from regular operation to energy harvesting, it is necessary to investigate ways to automatically align the USR with the wake of a bluff body from an arbitrary nearby initial position.

However, control strategies for USRs which do consider environmental disturbances such as ocean currents have mainly focused on following a path or trajectory despite the presence of the disturbance [77, 85, 86, 113]. In contrast, automatic alignment with a wake resulting from an unknown current is not easily achieved by following a prescribed path, but instead requires some other specification of the control objective. Such control approaches, in which the objective is to adapt a vehicle's behaviour to environmental disturbances rather than resist them, have been developed for traditional surface and underwater vehicles [25, 48, 81, 82, 123].

In this chapter, a control method for automatic alignment of a USR with a wake behind a bluff is developed drawing inspiration from the method of weather optimal position control (WOPC) [48, 81, 82, 123]. WOPC is a method for positioning a surface vessel so that the mean environmental force attacks through the centerline of the vessel, presented in [48, 123]. This method enables the vessel to align itself with the mean environmental force without measuring it, which can be difficult or impossible in practice. The method consists of designing a control law which keeps the position of the vessel on a circle of constant radius around a point on the ocean surface, and keeps it oriented toward the point. Thinking of the environmental force as a gravitational force and the controlled vessel as a pendulum, the environmental force pushes the vessel as far downstream as possible while remaining on the circle, where the controlled heading of the vessel will align against the environmental force. This concept is illustrated in Figure 8.1. By continuously updating the center of the circle, the position of the vehicle can also be kept fixed. If the center of the circle is fixed, the method is also referred to as weather optimal heading control (WOHC). It is shown that the method renders both the error in heading and the position of the vessel ES, and its efficacy is demonstrated both in simulations and experiments.

The method as presented in [48] assumes a fully actuated surface vessel. A USR,

on the other hand, can primarily only control its heading and tangential velocity while it swims. Furthermore, these quantities cannot be controlled directly, but must be controlled through the modulation of the periodic swimming pattern. As a result, neither heading nor velocity are asymptotically stabilizable to a reference, but rather to a vicinity of the reference. Furthermore, [48] assumes that the current consists solely of a component with a slowly-varying magnitude and a slowly-varying direction. While this is a reasonable simplification for a large ship, it is not necessarily as reasonable for a fairly light robot attempting to align itself in a Von Karman vortex street. In [82], WOPC is adapted to vessels which are unactuated in the sway direction. However, the vessel is still assumed to have direct control authority in surge and yaw. Additionally, while non-constant, slowly varying disturbances are explored in simulations, their effect on the stability is not explored in any stability analysis. Furthermore, more general time-varying components are not considered. In order to be applicable to USRs swimming in a wake, the control method must handle the presence of time-varying disturbances as well as be suitable for underactuated vehicles.

The strategy proposed in this chapter relies on position measurements relative to an object in the environment. This is beneficial in an underwater setting, where global position measurements can be difficult to obtain. Relative position measurements, on the other hand, may be possible to obtain using a computer vision system on board, or if the object is part of a known structure, it can be equipped with a beacon for localization.

8.2 Modelling and control of joints, velocity and heading

The overarching control objective considered in this chapter is to hold a desired distance to a bluff body while adapting to a current that may change direction over time. To achieve this, the joint angles of each link ϕ_i are controlled to follow a sinusoidal gait which has been shown to induce forward motion in [84]. The gait to be used here is referred to as the lateral undulation gait. The desired angle of each joint during this gait, for $i = 1, \dots, N_l - 1$ being the joint number, is given by

$$\phi_{d,i} = \alpha \gamma_1(i) \sin(\lambda + (i - 1)\delta) + \phi_0 \quad (8.1)$$

where α is the amplitude, $\gamma_1(i)$ is a scaling function that allows us to vary the amplitude along the snake body, δ is the phase shift between adjacent joints, and N_l is the number of links. The variables λ and ϕ_0 are the frequency and turning parameters. The joint reference signals are assembled into a vector $\phi_d \in \mathbb{R}^{N_l-1}$.

The overall control system consists of two components: the low-level controller and the high-level controller. The low-level controller's main objective is to generate and follow a sinusoidal gait that achieves a desired tangential velocity v_{ref} and a desired heading θ_d . This is accomplished by utilizing the second derivative of the gait frequency $\ddot{\lambda} = u_\lambda$ and the turning parameter $\phi_0 = u_\phi$ as virtual control inputs. The high-level controller generates the desired tangential velocity and heading, designed to achieve alignment of the USR with the wake. A block diagram of the complete

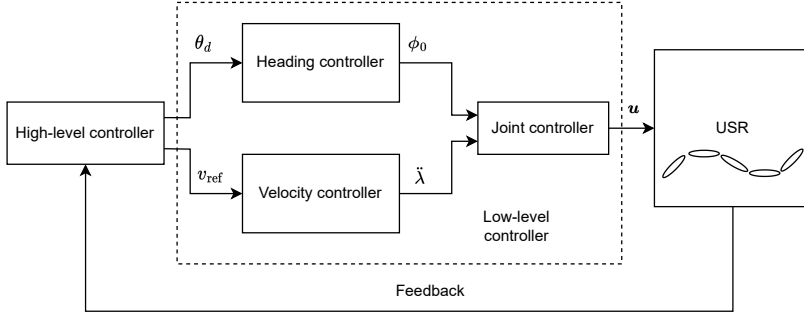


Figure 8.2: Block diagram of the control system

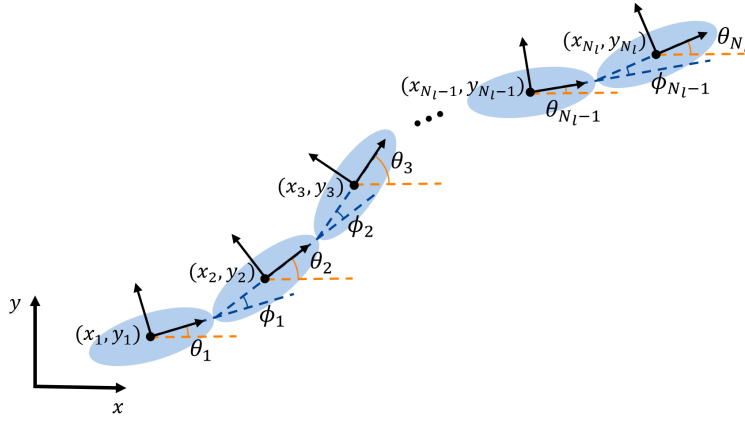


Figure 8.3: Illustration of the link frames, link angles and relative joint angles of a USR.

control system is given in Figure 8.2, where the low-level controller is split into three subcomponents: a joint controller as the lowest control level, and heading and velocity controllers. In this section, the low-level controller design is presented.

8.2.1 Model of USR dynamics

The equations of motion for a USR which are needed for the design of this control approach, are presented in this section. A detailed model of a USR, where the full kinematics and dynamics, of a planar snake robot with revolute joints are considered, is presented in [74]. The model from [74] includes expressions for the hydrodynamic effects as functions of velocities and accelerations of the USR. In the model used in this chapter, the expressions for the hydrodynamic effects will be somewhat simplified compared to [74].

The orientation and position of link i along the global x - and y -axes are defined as θ_i , x_i and y_i , respectively. This is illustrated Figure 8.3. Additionally, the position of the center of mass (CM) of the snake robot is defined as $\mathbf{p}_{\text{cm}} = [x_{\text{cm}}, y_{\text{cm}}]^\top$. The

hydrodynamical forces along the x - and y -axes of each link i , and moments acting on it, are defined as $f_{hx,i}$, $f_{hy,i}$ and $\tau_{h,i}$, respectively. Furthermore, define $\mathbf{F} = [\mathbf{F}_x, \mathbf{F}_y]^\top$, where $\mathbf{F}_x = [f_{hx,1}, f_{hx,2}, \dots, f_{hx,N_l}]^\top$ and $\mathbf{F}_y = [f_{hy,1}, f_{hy,2}, \dots, f_{hy,N_l}]^\top$. To model the hydrodynamical forces, a slowly moving USR where the linear hydrodynamical drag forces and resistive torque are dominant is considered. Therefore, the nonlinear drag forces, added mass forces, and added mass inertia torques can be disregarded. The hydrodynamical forces can then be modelled as

$$\mathbf{F} = \mathbf{F}_D = - \begin{bmatrix} c_t(\mathbf{C}_\theta)^2 + c_n(\mathbf{S}_\theta)^2 & (c_t - c_n)\mathbf{C}_\theta\mathbf{S}_\theta \\ (c_t - c_n)\mathbf{C}_\theta\mathbf{S}_\theta & c_t(\mathbf{S}_\theta)^2 + c_n(\mathbf{C}_\theta)^2 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{X}}_r \\ \dot{\mathbf{Y}}_r \end{bmatrix} \quad (8.2)$$

where $c_t, c_n > 0$ are drag coefficients in the tangential and normal direction of a link, respectively. The matrices $\mathbf{S}_\theta, \mathbf{C}_\theta$ are diagonal matrices given as

$$\mathbf{S}_\theta = \text{diag}(\sin(\theta_1), \dots, \sin(\theta_{N_l})) \quad (8.3a)$$

$$\mathbf{C}_\theta = \text{diag}(\cos(\theta_1), \dots, \cos(\theta_{N_l})) \quad (8.3b)$$

and

$$\dot{\mathbf{X}}_r = \dot{\mathbf{X}} - \mathbf{e}V_x, \quad (8.4a)$$

$$\dot{\mathbf{Y}}_r = \dot{\mathbf{Y}} - \mathbf{e}V_y, \quad (8.4b)$$

are the relative velocities of the links, with $\mathbf{X} = [x_1, \dots, x_{N_l}]^\top$, $\mathbf{Y} = [y_1, \dots, y_{N_l}]^\top$, and V_x, V_y being the current in the global x - and y -directions respectively. The resistive torque of the fluid is described similarly by $\boldsymbol{\tau}_{\text{hyd}} = -\mathbf{\Lambda}\boldsymbol{\theta}$, where $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_{N_l}]^\top$ and $\mathbf{\Lambda}$ is a diagonal matrix of drag coefficients. The equations of motion are

$$\mathbf{M}_\theta \ddot{\boldsymbol{\theta}} + \mathbf{W}_\theta \dot{\boldsymbol{\theta}}^2 + \mathbf{Q}_\theta(\mathbf{F}_D) + \boldsymbol{\tau}_{\text{hyd}} = \mathbf{D}^\top \mathbf{u}, \quad (8.5a)$$

$$\ddot{\mathbf{p}}_{\text{cm}} = \frac{1}{N_l m} \begin{bmatrix} \mathbf{e}^\top \mathbf{F}_{Dx} \\ \mathbf{e}^\top \mathbf{F}_{Dy} \end{bmatrix}, \quad (8.5b)$$

where

$$\mathbf{M}_\theta = J\mathbf{I}_{N_l} + ml^2\mathbf{S}_\theta\mathbf{V}\mathbf{S}_\theta + ml^2\mathbf{C}_\theta\mathbf{V}\mathbf{C}_\theta, \quad (8.6a)$$

$$\mathbf{W}_\theta = ml^2\mathbf{S}_\theta\mathbf{V}\mathbf{C}_\theta - ml^2\mathbf{C}_\theta\mathbf{V}\mathbf{S}_\theta, \quad (8.6b)$$

$$\mathbf{Q}_\theta(\mathbf{F}) = -l\mathbf{S}_\theta\mathbf{K}\mathbf{F}_{Dx} + l\mathbf{C}_\theta\mathbf{K}\mathbf{F}_{Dy}, \quad (8.6c)$$

$$\mathbf{V} = \mathbf{A}^\top(\mathbf{D}\mathbf{D}^\top)^{-1}\mathbf{A}, \quad (8.6d)$$

$$\mathbf{K} = \mathbf{A}^\top(\mathbf{D}\mathbf{D}^\top)^{-1}\mathbf{D}. \quad (8.6e)$$

The matrices with the subscript θ are dependent on $\boldsymbol{\theta}$, and the functional argument is omitted to conserve space. The length, mass and moment of inertia of the links are given by l, m and J , respectively. The vectors $\mathbf{e}$ and $\bar{\mathbf{e}}$ are summation vectors defined as $\mathbf{e} = [1, \dots, 1]^\top \in \mathbb{R}^{N_l}$ and $\bar{\mathbf{e}} = [1, \dots, 1]^\top \in \mathbb{R}^{N_l-1}$, and the matrix $\bar{\mathbf{D}} = \mathbf{D}^\top(\mathbf{D}\mathbf{D}^\top)^{-1}$. The matrices $\mathbf{A}, \mathbf{D} \in \mathbb{R}^{(N_l-1) \times N_l}$ are given by

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}. \quad (8.7)$$

The relative angle ϕ_i of each link can be found through the transformation

$$\bar{\phi} = \mathbf{H}^{-1}\boldsymbol{\theta}, \quad (8.8)$$

where $\bar{\phi} = [\phi_1, \dots, \phi_{N_l-1}, \theta_{N_l}]^\top$ and

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{N_l \times N_l}, \quad (8.9)$$

resulting in $\phi_i = \theta_i - \theta_{i+1}$. The position of the center of mass of each link can be found through the transformation

$$\mathbf{p} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} -l\mathbf{K}^\top \cos(\boldsymbol{\theta}) + \mathbf{e}x_{\text{cm}} \\ -l\mathbf{K}^\top \sin(\boldsymbol{\theta}) + \mathbf{e}y_{\text{cm}} \end{bmatrix}. \quad (8.10)$$

Furthermore, the heading of the USR is taken to be the average orientation of the links, that is

$$\bar{\theta} = \frac{1}{N_l} \sum_{i=1}^{N_l} \theta_i. \quad (8.11)$$

8.2.2 Joint controller

The objective of the joint controller is to generate control inputs that asymptotically stabilize the joint angles of the USR to the time-varying references given by the lateral undulation gait (8.1). To design a joint controller it is desirable to separate the directly actuated and indirectly actuated, i.e. unactuated, states. The states are defined as $\mathbf{q} = [\boldsymbol{\phi}^\top, \mathbf{q}_u^\top]^\top$, with the unactuated states $\mathbf{q}_u = [\theta_N, \mathbf{p}_{\text{cm}}^\top]^\top$ and actuated states $\boldsymbol{\phi} = [\phi_1, \dots, \phi_{N_l-1}]^\top$. This section follows the approach presented by [93] to separate the dynamics. Using the transformation between the relative link angles and global orientation (8.8), the model (8.5) can be rewritten as

$$\bar{\mathbf{M}}_{\bar{\phi}} \ddot{\bar{\phi}} + \bar{\mathbf{W}}_{\bar{\phi}} + \bar{\mathbf{Q}}_{\phi}(\mathbf{F}) + \bar{\mathbf{A}}(\boldsymbol{\tau}_{\text{hyd}}) = \bar{\mathbf{B}}\mathbf{u}, \quad (8.12)$$

where

$$\bar{\mathbf{M}}_{\bar{\phi}} = \begin{bmatrix} \mathbf{H}^\top \mathbf{M}_{H\bar{\phi}} \mathbf{H} & \mathbf{0}_{N_l \times 2} \\ \mathbf{0}_{2 \times N_l} & N_l m \mathbf{I}_{2 \times 2} \end{bmatrix}, \quad (8.13a)$$

$$\bar{\mathbf{W}}_{\bar{\phi}} = \begin{bmatrix} \mathbf{H}^\top \mathbf{W}_{H\bar{\phi}} \text{diag}(\mathbf{H}\dot{\bar{\phi}})\mathbf{H}\dot{\bar{\phi}} \\ \mathbf{0}_{2 \times 1} \end{bmatrix}, \quad (8.13b)$$

$$\bar{\mathbf{Q}}_{\phi}(\mathbf{F}_D) = \begin{bmatrix} -l\mathbf{H}^\top (\mathbf{S}_{H\bar{\phi}} \mathbf{K} \mathbf{F}_{Dx} - \mathbf{C}_{H\bar{\phi}} \mathbf{K} \mathbf{F}_{Dy}) \\ -\mathbf{e}^\top \mathbf{F}_{Dx} \\ -\mathbf{e}^\top \mathbf{F}_{Dy} \end{bmatrix}, \quad (8.13c)$$

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{I}_{(N_l-1) \times (N_l-1)} \\ \mathbf{0}_{3 \times N_l-1} \end{bmatrix}, \quad (8.13d)$$

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{H}^\top \boldsymbol{\tau}_{\text{hyd}} \\ \mathbf{0}_{2 \times 1} \end{bmatrix}. \quad (8.13e)$$

The subscript ϕ replaces the functional argument (ϕ) for the purpose of space conservation. The dynamics (8.12) can then be rewritten to

$$\bar{\mathbf{M}}_{11}\ddot{\phi} + \bar{\mathbf{M}}_{12}\ddot{\mathbf{q}}_u + \bar{\mathbf{W}}_1 + \mathbf{F}_1 + \boldsymbol{\tau}_1 = \mathbf{u}, \quad (8.14a)$$

$$\bar{\mathbf{M}}_{21}\ddot{\phi} + \bar{\mathbf{M}}_{22}\ddot{\mathbf{q}}_u + \bar{\mathbf{W}}_2 + \mathbf{F}_2 + \boldsymbol{\tau}_2 = \mathbf{0}_{3 \times 1}, \quad (8.14b)$$

where the matrices from (8.12) have been separated into submatrices, with $\bar{\mathbf{M}}_{11} \in \mathbb{R}^{(N_l-1) \times (N_l-1)}$, $\bar{\mathbf{M}}_{12} \in \mathbb{R}^{(N_l-1) \times 3}$, $\bar{\mathbf{M}}_{21} \in \mathbb{R}^{3 \times (N_l-1)}$ and $\bar{\mathbf{M}}_{22} \in \mathbb{R}^{3 \times 3}$. Additionally, the separated Coriolis and centripetal vectors are defined as $\bar{\mathbf{W}}_1 \in \mathbb{R}^{(N_l-1)}$, $\bar{\mathbf{W}}_2 \in \mathbb{R}^3$, respectively. Furthermore, the transformed hydrodynamical forces (8.13c) and moments (8.13e) are separated and denoted $\mathbf{F}_1 \in \mathbb{R}^{(N_l-1)}$, $\mathbf{F}_2 \in \mathbb{R}^3$ and $\boldsymbol{\tau}_1 \in \mathbb{R}^{(N_l-1)}$, $\boldsymbol{\tau}_2 \in \mathbb{R}^3$, respectively. To find an expression for $\ddot{\mathbf{q}}_u$, (8.14b) is rewritten to

$$\ddot{\mathbf{q}}_u = -\bar{\mathbf{M}}_{22}^{-1}(\bar{\mathbf{M}}_{21}\ddot{\phi} + \bar{\mathbf{W}}_2 + \mathbf{F}_2 + \boldsymbol{\tau}_2). \quad (8.15)$$

Note that $\bar{\mathbf{M}}_{22}^{-1}$ is well-defined as a consequence of the uniform positive definiteness of the complete system inertia matrix $\bar{\mathbf{M}}_\phi$ [93]. The unactuated states are substituted in (8.14a) by the expression (8.15), which results in

$$\begin{aligned} &(\bar{\mathbf{M}}_{11} - \bar{\mathbf{M}}_{12}\bar{\mathbf{M}}_{22}^{-1}\bar{\mathbf{M}}_{21})\ddot{\phi} + \bar{\mathbf{W}}_1 + \mathbf{F}_1 + \boldsymbol{\tau}_1 \\ &\quad - \bar{\mathbf{M}}_{12}\bar{\mathbf{M}}_{22}^{-1}(\bar{\mathbf{W}}_2 + \mathbf{F}_2 + \boldsymbol{\tau}_2) = \mathbf{u}. \end{aligned} \quad (8.16)$$

The following matrix is then defined:

$$\bar{\mathbf{M}}_2 = \bar{\mathbf{M}}_{11} - \bar{\mathbf{M}}_{12}\bar{\mathbf{M}}_{22}^{-1}\bar{\mathbf{M}}_{21}. \quad (8.17)$$

We will now use integrator backstepping to design the control input $\mathbf{u}$. Let ϕ_{ref} be a vector with elements given by (8.1), and let $\tilde{\phi} = \phi - \phi_{\text{ref}}$. Moreover, let $\mathbf{z}_\phi = \dot{\phi} - \dot{\zeta}_\phi$ where $\dot{\zeta}_\phi = \dot{\phi}_{\text{ref}} - \mathbf{K}_{\phi,1}\tilde{\phi}$, and $\mathbf{K}_{\phi,1}$ is a positive definite matrix. The control input is selected as

$$\begin{aligned} \mathbf{u} = &\bar{\mathbf{W}}_1 + \mathbf{F}_1 + \boldsymbol{\tau}_1 - \bar{\mathbf{M}}_{12}\bar{\mathbf{M}}_{22}^{-1}(\bar{\mathbf{W}}_2 + \mathbf{F}_2 + \boldsymbol{\tau}_2) \\ &+ \bar{\mathbf{M}}_2(\dot{\zeta}_\phi - \mathbf{K}_{\phi,2}\mathbf{z}_\phi - \ddot{\phi}), \end{aligned} \quad (8.18)$$

where $\mathbf{K}_{\phi,2}$ is a positive definite matrix. Inserting the controller (8.18) and defining a new error state $\boldsymbol{\varrho}_1 = [\tilde{\phi}, \mathbf{z}_\phi]^\top$, the closed-loop joint error dynamics become:

$$\dot{\boldsymbol{\varrho}}_1 = \begin{bmatrix} -\mathbf{K}_{\phi,1}\tilde{\phi} + \mathbf{z}_\phi \\ -\mathbf{K}_{\phi,2}\mathbf{z}_\phi - \dot{\phi} \end{bmatrix}. \quad (8.19)$$

Proposition 8.1. *The system described by (8.19) is GES at the origin.*

Proof. Consider the Lyapunov function $V_\phi = \frac{1}{2}\boldsymbol{\varrho}_1^\top \boldsymbol{\varrho}_1$. Differentiating the Lyapunov function and inserting for $\dot{\zeta}_\phi$ and $\mathbf{z}_\phi$ gives

$$\dot{V}_\phi \leq -\mathbf{K}_{\phi,1}\tilde{\phi}^2 - \mathbf{K}_{\phi,2}\mathbf{z}_\phi^2 \quad (8.20)$$

By [78, Theorem 4.10], the origin of the system is GES. $\square$

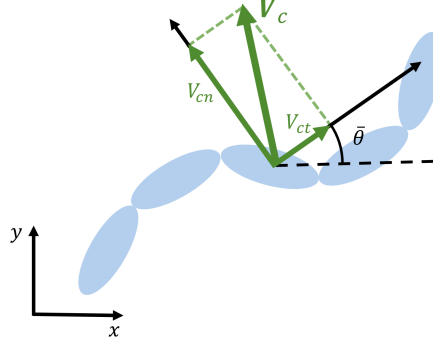


Figure 8.4: Illustration of the current decomposed in the body frame of the USR.

8.2.3 Simplified model of a USR following a swimming gait

The complexity of the model presented in Section 8.2.1 makes it challenging to use for design of heading and velocity controllers for the USR as a whole. Therefore, a control-oriented model was developed for USRs exposed to currents [83], approximating the revolute joint motion with prismatic joints with the aim to capture the qualitative behaviour of the complex model. This control-oriented model assumes that the currents are time-invariant and irrotational. However, despite being based on a simplification of the joints, it still includes cross-terms between states, reflecting the intricate dependencies between the link motion and tangential and normal velocities. To achieve reference tracking for the heading and tangential velocity and develop controllers for this control-oriented model, it is often necessary to cancel these terms. Furthermore, the currents in a wake behind a bluff body are highly time-varying and rotational. Therefore, the parameters identified for the hydrodynamical drag and resistive torques [83] will likely not give a good approximation of the hydrodynamical forces that affect the USR during operations in a wake. As discussed in [115], attempts to cancel poorly approximated terms, may result in degraded performance.

Therefore, a new simplified model is derived here based on treating the whole USR is considered a single rigid body during sinusoidal motion. Parameters of the gait reference (8.1) will now act as control inputs, with $\ddot{\lambda} = u_{\lambda}$ and $\phi_0 = u_{\phi}$ being control inputs for the tangential velocity and angular motion, respectively. The current velocity in the global frame is denoted as $\mathbf{V}_c(t)$. The tangential and normal velocity components of the USR and current, given in the body frame of the USR, are defined as $v_t, v_n, V_{ct}(t)$ and $V_{cn}(t)$, respectively. This decomposition of the current is illustrated in Figure 8.4. The relation between $\mathbf{V}_c(t)$, $V_{ct}(t)$ and $V_{cn}(t)$ is:

$$\begin{bmatrix} V_{ct}(t) \\ V_{cn}(t) \end{bmatrix} = \mathbf{R}(\bar{\theta})^{\top} \mathbf{V}_c(t) \quad (8.21)$$

where $\mathbf{R}(\cdot)$ denotes the two-dimensional rotation matrix:

$$\mathbf{R}(x) = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix} \quad (8.22)$$

The following assumptions are made:

Assumption 8.1. The USR is following the sinusoidal gait given by (8.1).

Assumption 8.1 constitutes the premise for constructing a simplified model of the dynamics of the USR as a whole while its motion follows a certain pattern, namely gaits which can be described by the equation (8.1). A USR with no other actuators than those of its joints is dependent on following some motion pattern to produce any forward locomotion [121]. While other gaits resulting in forward motion than those encompassed by (8.1) may exist, in this chapter we focus on those described by (8.1) which encompasses a spectrum of bio-inspired gaits, including snake- and eel-like motion.

Assumption 8.2. The USR is moving against the current such that $(v_t - V_{ct}(t)) > 0$.

Assumption 8.3. The current velocity $\mathbf{V}_c(t)$ and its derivative are bounded by $V_M, V_{M,d} > 0$ such that $\|\mathbf{V}_c(t)\| \leq V_M, \|\dot{\mathbf{V}}_c(t)\| \leq V_{M,d}$.

As seen from the control-oriented model [83], the USR can only turn while the relative tangential velocity is non-zero. Therefore, Assumption 8.2 ensures that the USR can turn. Because the objective of this controller is to align the USR with the wake of a bluff body, the USR will likely operate such that it is moving against the current, in which case Assumption 8.2 is satisfied. Note that control authority is only held over the turning rate of the USR and the tangential velocity v_t . The control objective is to stabilize the heading dynamics and the absolute tangential velocity dynamics to their references. Since there is no control authority over the velocity of the USR in the normal direction, it can at best be expected to consist of only a small contribution from the swimming motion, such that it ends up approximately following the current velocity. This is equivalent to the relative normal velocity being close to 0. To this end, the absolute tangential velocity v_t and the normal velocity relative to the ocean current $v_{n,\text{rel}} = v_n - V_{cn}(t)$ instead of v_n , are analysed. The resulting simplified model is given by

$$\ddot{\bar{\theta}} = -\lambda_\theta \dot{\bar{\theta}} + f_\theta \phi_0 + d_\theta(\tilde{\phi}) + d_1(t), \quad (8.23a)$$

$$\dot{v}_t = -\lambda_t(v_t - V_{ct}(t)) + f_t \ddot{\lambda} + d_t(\tilde{\phi}) + d_2(t) \quad (8.23b)$$

$$\dot{v}_{n,\text{rel}} = -\lambda_n v_{n,\text{rel}} + d_n(\tilde{\phi}) + d_3(t) - \dot{V}_{cn}(t), \quad (8.23c)$$

where $f_t > 0$ and $f_\theta > 0$. Furthermore, $\lambda_\theta, \lambda_t, \lambda_n > 0$ are the drag coefficients for the average angular velocity, tangential velocity, and normal velocity, respectively. The unmodeled dynamics and time-varying disturbances are given by $d_1(t)$, $d_2(t)$ and $d_3(t)$, and are assumed to be bounded:

Assumption 8.4. For any $\Delta > 0$ such that $\left\| \begin{bmatrix} \bar{\theta}, \dot{\bar{\theta}}, v_t, v_{n,\text{rel}} \end{bmatrix}^\top \right\| \leq \Delta$, there exists a bound $d_\Delta > 0$ such that $|d_1(t)| + |d_2(t)| + |d_3(t)| \leq d_\Delta$.

Disturbances that result from the USR deviating from the sinusoidal gait (8.1) are given by $d_\theta(\cdot)$, $d_t(\cdot)$ and $d_n(\cdot)$, which are gathered into a vector $\mathbf{d}_\phi(\tilde{\phi}) = \begin{bmatrix} d_\theta(\tilde{\phi}), d_t(\tilde{\phi}), d_n(\tilde{\phi}) \end{bmatrix}^\top$. The functions are continuous and satisfy $\mathbf{d}_\phi(0) = 0$. Additionally, the following assumption is made:

Assumption 8.5. The disturbances that result from deviation from the sinusoidal gait (8.1) are bounded such that $\|\mathbf{d}_\phi(\tilde{\phi})\| \leq \alpha_\phi(\|\tilde{\phi}\|)$, where $\alpha_\phi(\cdot)$ is a class $\mathcal{K}$ -function.

Remark. The unmodeled dynamics encompassed by $d_1(t), d_2(t), d_3(t)$ are comprised of various phenomena, including nonlinear drag, added mass effects, and cross-terms through which the states influence each other due to effects arising from drag and the sinusoidal motion. At lower velocities and accelerations, linear drag tends to dominate over the contributions of nonlinear drag and added mass. Additionally, the average impact of the cross-terms stemming from sinusoidal motion diminishes significantly when the turning offset ϕ_0 remains small. Moreover, for bounded velocities and accelerations, the hydrodynamical terms remain within bounds as well. In addition, it is reasonable to assume that the rate of change of the current velocities is bounded. Hence, as stated in Assumption 8.4, the hydrodynamical forces and other disturbances remain bounded. Furthermore, from Proposition 8.1 it follows that the joint error $\tilde{\phi}$ is bounded and converges exponentially to zero. Thus the validity of Assumption 8.5 is reinforced.

8.2.4 Velocity and heading controllers

Let $v_{\text{ref}}, \theta_{\text{ref}}$ be references for the tangential (i.e. forward) velocity and heading, respectively. It is assumed that the references and their derivatives are bounded. The objective of this low-level controller is to track the desired heading and tangential velocity. Meanwhile, the unactuated normal velocity will approach that of the ocean current. For the heading, a backstepping approach is utilized to design the control input u_ϕ . To this end, the error $z_1 = \bar{\theta} - \theta_{\text{ref}}, z_2 = \dot{\bar{\theta}} - \dot{\zeta}_1$, are defined, where $\dot{\zeta}_1$ is a reference angular velocity and serves as a virtual control input. The derivatives of z_1 and z_2 are given by

$$\begin{aligned} \dot{z}_1 &= \dot{\zeta}_1 - \dot{\theta}_{\text{ref}} + z_2 \\ \dot{z}_2 &= -\lambda_\theta \dot{\bar{\theta}} + f_\theta \phi_0 + d_\theta(\tilde{\phi}) + d_1(t) - \dot{\zeta}_1. \end{aligned} \quad (8.24)$$

In addition, the velocity error variable is defined as $\tilde{v}_t = v_t - v_{\text{ref}}$. Recalling that $\phi_0 = u_\phi$ and $\ddot{\lambda} = u_\lambda$, the reference angular velocity $\dot{\zeta}_1$ control inputs u_ϕ, u_λ are selected as

$$\dot{\zeta}_1 = \dot{\theta}_{\text{ref}} - k_{\theta,1} z_1 \quad (8.25a)$$

$$u_\phi = -k_{\theta,2} z_2 + \frac{1}{f_\theta} \dot{\zeta}_1 + \frac{\lambda_\theta}{f_\theta} \dot{\zeta}_1 - \frac{1}{f_\theta} z_1 \quad (8.25b)$$

$$u_\lambda = \frac{\lambda_t}{f_t} v_{\text{ref}} - k_v \tilde{v}_t + \dot{v}_{\text{ref}}, \quad (8.25c)$$

where the control gains are given by $k_{\theta,1} > 0$, $k_{\theta,2} > 0$ and $k_v > 0$. Inserting (8.25) into (8.23b), and (8.24) gives, together with (8.23c), the closed-loop dynamics

$$\dot{z}_1 = -k_{\theta,1}z_1 + z_2 \quad (8.26a)$$

$$\dot{z}_2 = -(f_\theta k_{\theta,1} + \lambda_\theta)z_2 - z_1 + d_1(t) + d_\theta(\tilde{\phi}) \quad (8.26b)$$

$$\dot{\tilde{v}}_t = -(f_t k_v + \lambda_t)\tilde{v}_t + d_2(t) + \lambda_t V_{ct}(t) + d_t(\tilde{\phi}) \quad (8.26c)$$

$$\begin{aligned} \dot{v}_{n,\text{rel}} &= -\lambda_n v_{n,\text{rel}} + d_n(\tilde{\phi}) + d_3(t) \\ &\quad + (z_2 + \dot{\theta}_d - k_{\theta,1}z_1) [\cos(\bar{\theta}) \quad \sin(\bar{\theta})] \mathbf{V}_c(t) \\ &\quad - [-\sin(\bar{\theta}) \quad \cos(\bar{\theta})] \dot{\mathbf{V}}_c(t), \end{aligned} \quad (8.26d)$$

which can be written as

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{\tilde{v}}_t \\ \dot{v}_{n,\text{rel}} \end{bmatrix} &= \overbrace{\begin{bmatrix} -k_{\theta,1} & 1 & 0 & 0 \\ -1 & -(f_\theta k_{\theta,1} + \lambda_\theta) & 0 & 0 \\ 0 & 0 & -(f_t k_v + \lambda_t) & 0 \\ -k_{\theta,1} V_{ct}(t) & V_{ct}(t) & 0 & -\lambda_n \end{bmatrix}}^{=\mathbf{A}_l(t)} \begin{bmatrix} z_1 \\ z_2 \\ \tilde{v}_t \\ v_{n,\text{rel}} \end{bmatrix} \\ &\quad + \underbrace{\begin{bmatrix} 0 \\ d_1(t) \\ d_2(t) + \lambda_t V_{ct}(t) \\ d_3(t) + \dot{\theta}_d V_{ct}(t) - [-\sin(\bar{\theta}) \quad \cos(\bar{\theta})] \dot{\mathbf{V}}_c(t) \end{bmatrix}}_{=\mathbf{d}(t)} + \begin{bmatrix} 0 \\ d_\theta(\tilde{\phi}) \\ d_t(\tilde{\phi}) \\ d_n(\tilde{\phi}) \end{bmatrix} \end{aligned} \quad (8.27)$$

The complete system, consisting of joint, heading and velocity controllers, can be considered a cascaded system where the driving system is given by (8.19) and the driven system is (8.27). Furthermore, the parameter vector is defined as $\boldsymbol{\vartheta}_l = [k_{\theta,1}, k_{\theta,2}, k_v, \lambda_n]^\top$, which belongs to $\Theta_l = \mathbb{R}_{>0}^4$. The following proposition is made:

Proposition 8.2. *The cascaded system (8.19) and (8.27) is UPAS on the set of admissible parameters Θ_l for bounded references.*

Proof. First, the nominal part of (8.27) is considered,

$$\dot{\mathbf{e}}_l = \mathbf{A}_l(t)\mathbf{e}_l + \mathbf{d}(t). \quad (8.28)$$

We choose the Lyapunov function candidate

$$\mathcal{V}_l = \frac{1}{2} \mathbf{e}_l^\top \mathbf{e}_l. \quad (8.29)$$

Differentiating $\mathcal{V}_l$ gives

$$\begin{aligned} \dot{\mathcal{V}}_l &= -k_{\theta,1}z_1^2 - (f_\theta k_{\theta,1} + \lambda_\theta)z_2^2 - (f_t k_v + \lambda_t)\tilde{v}_t^2 \\ &\quad - \lambda_n v_{n,\text{rel}}^2 + v_{n,\text{rel}}(z_2 - k_{\theta,1}z_1) V_{ct}(t) + \mathbf{e}_l^\top \mathbf{d}(t). \end{aligned} \quad (8.30)$$

Consider now the bounded local domain. For bounded states $\mathbf{e}_l$, since v_{ref} , θ_{ref} , $\dot{\theta}_{\text{ref}}$ are bounded by assumption, $\bar{\theta}$, $\dot{\bar{\theta}}$ and v_t , are also bounded. Then by Assumption 8.4

there exists a bound d_Δ such that $|d_1(t)| + |d_2(t)| + |d_3(t)| \leq d_\Delta$. Furthermore, from Assumption 8.3, the current velocities are bounded. Finally, by using Young's inequality on the term $v_{n,\text{rel}}(z_2 - k_{\theta,1}z_1)V_{ct}(t)$, the following bound can be found:

$$\begin{aligned} \dot{V}_l \leq & -k_{\theta,1} \left(1 - \frac{V_M \gamma_1}{2}\right) z_1^2 - \left((f_\theta k_{\theta,2} + \lambda_\theta) - \frac{V_M \gamma_2}{2}\right) z_2^2 \\ & - (f_t k_v + \lambda_t) \tilde{v}_t^2 - \left(\lambda_n - \frac{k_{\theta,1} V_M}{2\gamma_1} - \frac{V_M}{2\gamma_2}\right) v_{n,\text{rel}}^2 \\ & + \left(d_\Delta + \lambda_t V_M + V_{M,d} + |\dot{\theta}_{\text{ref}}| V_M\right) \|e_l\|. \end{aligned} \quad (8.31)$$

with $\gamma_1, \gamma_2 > 0$. The bound (8.31) is negative for $\|e_l\| > \frac{B_2}{B_1(\vartheta_l)}$, where

$$B_1(\vartheta_l) = \min \left(k_{\theta,1} \left(1 - \frac{V_M \gamma_1}{2}\right), (f_t k_v + \lambda_t), \right. \\ \left. ((f_\theta k_{\theta,2} + \lambda_\theta) - \frac{V_M \gamma_2}{2}), \left(\lambda_n - \frac{k_{\theta,1} V_M}{2\gamma_1} - \frac{V_M}{2\gamma_2}\right) \right), \quad (8.32a)$$

$$B_2(\vartheta_l) = d_\Delta + V_{M,d} + \lambda_t V_M + |\dot{\theta}_{\text{ref}}|_{\max} V_M, \quad (8.32b)$$

where $B_1(\cdot)$ can be increased or decreased by adjusting the parameters ϑ_l .

Hence Assumption 7.3 is satisfied, and by Assumption 8.5, Assumption 7.1 is also satisfied. Finally, since the driving system (8.19) is GES which is a stronger property than UPAS, the conditions of Theorem 7.2 are satisfied and the cascaded system (8.19), (8.27) is UPAS on the set of admissible parameters Θ_l . $\square$

8.3 Modelling and control of position in a wake

The high-level control takes its inspiration from [48, 82]. However, in both of these papers a virtual point is chosen for the vessel to revolve around, and the point must subsequently be updated in order for the vessel to stay in one spot while aligning its heading against the disturbance. However, in this chapter, the ultimate objective is to ensure that the USR ends up in the wake of a bluff body and tracks slow changes in the direction of this wake automatically. Firstly, as the system in question is underwater, the mean environmental force in [48] is replaced by an ocean current, and will be treated at the kinematic level. Secondly, it is assumed that the ocean current can be described by a mean component and its direction, both of which are constant or slowly-varying, onto which a smaller, general time-varying component is added. This smaller component is by definition zero-mean, as it is added to the mean of the current. The vortex street of a bluff body is generated exactly downstream of the bluff body with respect to the current, so that the centerline of the vortex street is parallel with the current direction. Thus, stabilizing the position of the USR to a vicinity of the vortex street centerline is equivalent to stabilizing it to a vicinity of the downstream direction from the bluff body. Finally, note that the point around which the USR should revolve is constant and given by the centroid of the bluff body, in order to end up downstream from it in a similar way to the surface vessel in [48]. Thus, only WOHC, where the position of the hinge point is kept constant,

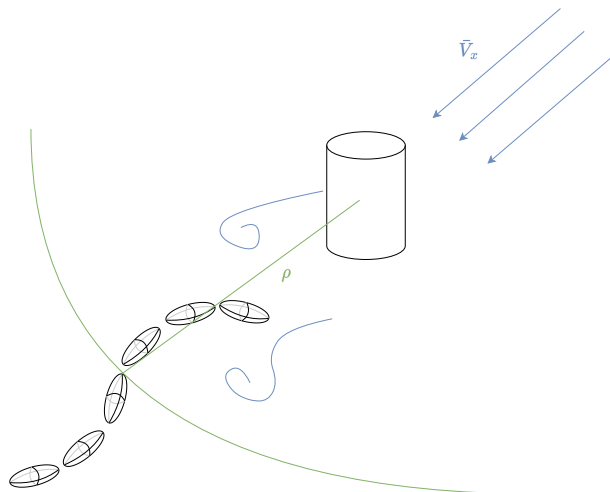


Figure 8.5: Illustration of USR operating in the wake behind a cylinder.

is relevant as a starting point for the method presented in this chapter. The concept is illustrated in Figure 8.5.

This section introduces control laws which generate heading and tangential velocity references for the USR. These control laws are designed to point the USR towards the bluff body and make its position converge to a circle of prescribed radius. Moreover, by employing Lyapunov analysis it is shown that this choice of references renders the position of the USR UPAS, provided that the references are followed sufficiently well by low-level controllers. This is the case when using the low-level controllers from Section 8.2, by Proposition 8.2.

8.3.1 Kinematics

In [48], the total environmental disturbance is modelled as a force acting on a surface vessel, and consists of the total influence from wind, waves and current. In the absence of wind as is the case for an underwater robot, the disturbance from the current can be modelled as a velocity.

Remark. Modelling the disturbance as a velocity rather than a force influencing the dynamics of the robot corresponds to assuming that the USR is swept along with the changes in the current instantaneously. This would be the case for a very light system. Comparing this to a linear drag model, the effect of the current on the robot velocity would be of first order, with the robot dynamics acting as a low-pass filter. Thus, the model used here represents a worst-case scenario with respect to the influence of time-varying current components on the robot velocity. The benefit of this approach is that the analysis becomes more modularized, so that the results from the analysis of the high-level controller are applicable to any underwater system equipped with controllers for heading and tangential velocity.

The derivative of the position of the CM of the USR, given in terms of the absolute tangential velocity v_t and relative normal velocity $v_{n,\text{rel}}$ of the USR is given by

$$\begin{aligned}\dot{\mathbf{p}}_{\text{cm}} &= \mathbf{R}(\bar{\theta}) \begin{bmatrix} v_t \\ v_{n,\text{rel}} + V_{cn}(t) \end{bmatrix} \\ &= \mathbf{R}(\bar{\theta}) \left(\begin{bmatrix} v_t \\ v_{n,\text{rel}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{R}^\top(\bar{\theta}) \mathbf{V}_c(t) \right)\end{aligned}\quad (8.33)$$

where $\mathbf{V}_c(t)$ is the time-varying ocean current given in the global frame. The current $\mathbf{V}_c(t)$ can be described as

$$\mathbf{V}_c(t) = \mathbf{R}(\beta_c) \left(\begin{bmatrix} \bar{V}_x \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{V}_x(t) \\ \tilde{V}_y(t) \end{bmatrix} \right) \quad (8.34)$$

where β_c and $\bar{V}_x$ are constant or slowly varying, and the time-varying components $\tilde{V}_x(t)$, $\tilde{V}_y(t)$ satisfy $|\tilde{V}_x(t)| \leq \tilde{V}_{x,\text{max}}$, $|\tilde{V}_y(t)| \leq \tilde{V}_{y,\text{max}}$ for some $\tilde{V}_{x,\text{max}}, \tilde{V}_{y,\text{max}} > 0$, and $\tilde{V}_{x,\text{max}} < \bar{V}_x$.

Remark. Having a somewhat steady current direction is necessary for the formation of a proper Von Karman vortex street. Thus, under conditions where there exists a vortex street for the USR to align with, it is reasonable to assume that β_c is constant.

Remark. In the case of a wake behind an object, a more precise description of the current would be as a time-varying vector field $\mathbf{V}_c(t, \mathbf{p})$, which is periodic at a fixed point $\mathbf{p}$ in the case of a steady wake. In the case of a current described by a vector field, $\mathbf{V}_c(t) = \mathbf{V}_c(t, \mathbf{p}(t))$ can be treated as the current along a trajectory $\mathbf{p}(t)$ of the system. Directly behind the cylinder, the average of $\tilde{V}_y(t)$ will be 0 over one period.

As in [48], the position of the robot is described in polar coordinates, but the origin is placed at a fixed object (e.g. a cylinder). The objective is to have the position of the snake robot converge to a desired constant distance ρ_d downstream from the object.

Let ρ be the radius and γ the angle of the polar coordinates, so that

$$\begin{bmatrix} x_{\text{cm}} \\ y_{\text{cm}} \end{bmatrix} = \begin{bmatrix} \rho \cos(\gamma) \\ \rho \sin(\gamma) \end{bmatrix}. \quad (8.35)$$

By differentiating ρ , γ , the following kinematic relation can be derived:

$$\begin{bmatrix} \dot{\rho} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \mathbf{R}^\top(\gamma) \mathbf{R}(\bar{\theta}) \left(\begin{bmatrix} v_t \\ v_{n,\text{rel}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{R}^\top(\bar{\theta}) \mathbf{V}_c(t) \right) \quad (8.36)$$

for $\rho \neq 0$.

Remark. Since the origin is at the position of an object in the environment, $\rho = 0$ is highly undesirable in any case, as it would lead to a collision with the object. Thus, requiring $\rho \neq 0$ is not restrictive for a practical application, and the analysis shows that it can also be avoided through a proper choice of initial states and parameters.

8.3.2 Heading and velocity references

As in [48], the heading reference is chosen to be inwards towards the origin, i.e.

$$\theta_{\text{ref}} = \gamma + \pi. \quad (8.37)$$

Inserting $v_t = v_{\text{ref}} + \tilde{v}_t$, $\bar{\theta} = \theta_{\text{ref}} + \tilde{\theta}$, (8.37) and (8.34), (8.36) can be written as

$$\begin{aligned} \begin{bmatrix} \dot{\rho} \\ \dot{\gamma} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \left(-\mathbf{R}(\tilde{\theta}) \begin{bmatrix} v_{\text{ref}} + \tilde{v}_t \\ v_{n,\text{rel}} \end{bmatrix} \right. \\ &\quad \left. + \mathbf{R}(\tilde{\theta}) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{R}^\top(\tilde{\theta}) \mathbf{R}^\top(\gamma - \beta_c) \begin{bmatrix} \bar{V}_x + \tilde{V}_x \\ \tilde{V}_y \end{bmatrix} \right). \end{aligned} \quad (8.38)$$

Define the error variables $\tilde{\gamma} = \gamma - \beta_c$, and $\tilde{\rho} = \rho - \rho_d$. Setting $\dot{\beta}_c = 0$, the resulting error dynamics are given by

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{\rho}} \\ \dot{\tilde{\gamma}} \end{bmatrix} &= \begin{bmatrix} -v_{\text{ref}} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\rho_d + \tilde{\rho}} \end{bmatrix} \left(\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \mathbf{T}(\tilde{\theta}) \right) \mathbf{R}^\top(\tilde{\gamma}) \begin{bmatrix} \bar{V}_x + \tilde{V}_x \\ \tilde{V}_y \end{bmatrix} \right. \\ &\quad \left. + (I - \mathbf{R}(\tilde{\theta})) \begin{bmatrix} v_{\text{ref}} \\ 0 \end{bmatrix} - \mathbf{R}(\tilde{\theta}) \begin{bmatrix} \tilde{v}_t \\ v_{n,\text{rel}} \end{bmatrix} \right) \end{aligned} \quad (8.39)$$

where

$$\mathbf{T}(\tilde{\theta}) = \begin{bmatrix} \sin^2(\tilde{\theta}) & -\sin(\tilde{\theta})\cos(\tilde{\theta}) \\ -\sin(\tilde{\theta})\cos(\tilde{\theta}) & -\sin^2(\tilde{\theta}) \end{bmatrix}. \quad (8.40)$$

The reference velocity, v_{ref} , is chosen as

$$v_{\text{ref}} = k_\rho \cos(\tilde{\theta}) \tilde{\rho}. \quad (8.41)$$

As in [82], the error term is scaled by $\cos(\tilde{\theta})$ to avoid unnecessary and undesired control effort while the heading is not as desired. This corresponds to using only the part of the error that is parallel to the body x -axis of the system, in other words the error in the actuated direction.

Analysis of closed-loop system

The dynamics (8.39) in closed loop with (8.37), (8.41) become

$$\begin{aligned} \dot{\tilde{\rho}} &= -k_\rho \tilde{\rho} + \left(s\tilde{\theta}^2 c\tilde{\gamma} + s\tilde{\theta} c\tilde{\theta} s\tilde{\gamma} \right) \left(\bar{V}_x + \tilde{V}_x \right) \\ &\quad + \left((s\tilde{\theta})^2 s\tilde{\gamma} - s\tilde{\theta} c\tilde{\theta} c\tilde{\gamma} \right) \tilde{V}_y + s\tilde{\theta}^2 k_\rho \tilde{\rho} - c\tilde{\theta} \tilde{v}_t + s\tilde{\theta} v_{n,\text{rel}} \end{aligned} \quad (8.42a)$$

$$\begin{aligned} \dot{\tilde{\gamma}} &= \frac{1}{\tilde{\rho} + \rho_d} \left(-s\tilde{\gamma} \left(\bar{V}_x + \tilde{V}_x \right) + \left((s\tilde{\theta})^2 s\tilde{\gamma} - s\tilde{\theta} c\tilde{\theta} c\tilde{\gamma} \right) \left(\bar{V}_x + \tilde{V}_x \right) \right. \\ &\quad \left. + \left((s\tilde{\theta})^2 c\tilde{\gamma} + s\tilde{\theta} c\tilde{\theta} s\tilde{\gamma} \right) \tilde{V}_y - s\tilde{\theta} \left(k_\rho c\tilde{\theta} \tilde{\rho} + \tilde{v}_t \right) - c\tilde{\theta} v_{n,\text{rel}} + c\tilde{\gamma} \tilde{V}_y \right) \end{aligned} \quad (8.42b)$$

where $c\tilde{\gamma}$, $s\tilde{\gamma}$ denote $\cos \tilde{\gamma}$, $\sin \tilde{\gamma}$, respectively, and similarly for $\tilde{\theta}$, for the sake of brevity.

Let $\boldsymbol{\xi} = [\tilde{v}_t, v_{n,\text{rel}}, \tilde{\theta}, \dot{\tilde{\theta}}, \boldsymbol{\varrho}_1^\top]^\top$, and let $k_\rho, \vartheta_\gamma = \frac{\tilde{V}_{y,\text{max}}}{\tilde{V}_x - \tilde{V}_{x,\text{max}}}$ be parameters of the system. The system (8.42) may then be rewritten as

$$\dot{\tilde{\gamma}} = f_1(t, \tilde{\rho}, \tilde{\gamma}, \vartheta_\gamma) + \mathbf{g}_1(t, \tilde{\rho}, \tilde{\gamma}, \boldsymbol{\xi}, k_\rho) \boldsymbol{\xi} \quad (8.43a)$$

$$\dot{\tilde{\rho}} = f_2(\tilde{\rho}, k_\rho) + \mathbf{g}_2(t, \tilde{\rho}, \tilde{\gamma}, \boldsymbol{\xi}, k_\rho) \boldsymbol{\xi} \quad (8.43b)$$

where

$$f_1 = \frac{1}{\tilde{\rho} + \rho_d} \left(-s\tilde{\gamma} (\tilde{V}_x + \tilde{V}_x) + c\tilde{\gamma}\tilde{V}_y \right) \quad (8.43c)$$

$$\mathbf{g}_1 = \frac{1}{\tilde{\rho} + \rho_d} \begin{bmatrix} -s\tilde{\theta}, & -c\tilde{\theta}, & \frac{s\tilde{\theta}}{\tilde{\theta}} g_{13}, & 0, & \mathbf{0}_{1 \times 2(N_l-1)} \end{bmatrix} \quad (8.43d)$$

$$g_{13} = \left(-c\tilde{\theta}k_\rho\tilde{\rho} + \left(s\tilde{\theta}s\tilde{\gamma} - c\tilde{\theta}c\tilde{\gamma} \right) (\tilde{V}_x + \tilde{V}_x) - \left(s\tilde{\theta}c\tilde{\gamma} + c\tilde{\theta}s\tilde{\gamma} \right) \tilde{V}_y \right) \quad (8.43e)$$

$$f_2 = -k_\rho\tilde{\rho} \quad (8.43f)$$

$$\mathbf{g}_2 = \begin{bmatrix} -c\tilde{\theta}, & s\tilde{\theta}, & \frac{s\tilde{\theta}}{\tilde{\theta}} g_{23}, & 0, & \mathbf{0}_{1 \times 2(N_l-1)} \end{bmatrix} \quad (8.43g)$$

$$g_{23} = \left(s\tilde{\theta}k_\rho\tilde{\rho} + \left(s\tilde{\theta}c\tilde{\gamma} + c\tilde{\theta}s\tilde{\gamma} \right) (\tilde{V}_x + \tilde{V}_x) + \left(s\tilde{\theta}s\tilde{\gamma} - c\tilde{\theta}c\tilde{\gamma} \right) \tilde{V}_y \right) \quad (8.43h)$$

The parameters ϑ_γ and k_ρ belong to the sets $\Theta_1 = \mathbb{R}_{\geq 0}$ and $\Theta_2 = \mathbb{R}_{> 0}$, respectively.

Proposition 8.3. *Let $\boldsymbol{\xi}$ be the state variable of a system that is UPAS on some parameter set Θ_3 . Then the system (8.43) is UPAS on $\Theta_1 \times \Theta_2 \times \Theta_3$.*

Corollary 8.1. *The cascaded system (8.19), (8.27) and (8.43) is UPAS on the parameter set $\Theta_1 \times \Theta_2 \times \Theta_l$.*

Proof of Proposition 8.3. The system (8.43) will be treated as a cascade, where first practical stability of (8.43b) is established, and consequently that ρ can be bounded away from 0 such that the dynamics of $\tilde{\gamma}$ are valid. Then, Theorem 7.2 is applied twice in succession to establish UPAS of the complete system.

Since the original variables ρ, γ are polar coordinates, ρ must be greater than zero, meaning that the dynamics (8.43) are only valid for $\tilde{\rho} \in (-\rho_d, \infty)$. Considering first the nominal part of (8.43b), $\dot{\tilde{\rho}} = f_2(\tilde{\rho}, k_\rho)$, it follows that since $k_\rho > 0$, the nominal dynamics are locally exponentially stable (LES). Consequently, by [78, Theorem 4.12] there exists a Lyapunov function candidate for the system (8.43b) which satisfies Assumption 7.3. Since $\tilde{V}_x, \tilde{V}_x, \tilde{V}_y$ are bounded, $\mathbf{g}_2$ satisfies Assumption 7.1, and $\boldsymbol{\xi}$ is the state of a system which is UPAS by assumption, by Theorem 7.2, the system (8.43b) in cascade with the dynamics of $\boldsymbol{\xi}$ is UPAS on $\Theta_2 \times \Theta_3$.

Since the system (8.43b) in cascade with the dynamics of $\boldsymbol{\xi}$ is UPAS, by choice of $\tilde{\rho}(t_0)$ and parameters, it can be ensured that $|\tilde{\rho}(t)| < \rho_d$ s.t. $\rho(t) = \tilde{\rho}(t) + \rho_d \geq \rho_{\min} > 0 \forall t \geq t_0$. Consequently, $\mathbf{g}_1$ satisfies Assumption 7.1. Consider then the nominal part of the dynamics (8.43a), where $\tilde{\rho}$ is treated as time-varying signal. The following Lyapunov function candidate is chosen:

$$V_\gamma(\tilde{\gamma}) = \frac{1}{2}\tilde{\gamma}^2, \quad (8.44)$$

which gives

$$\begin{aligned}
 \frac{\partial V_\gamma}{\partial t} + \frac{\partial V_\gamma}{\partial \tilde{\gamma}} f_2(\cdot) &= \frac{1}{\tilde{\rho} + \rho_d} \tilde{\gamma} \left(-s\tilde{\gamma} \left(\bar{V}_x + \tilde{V}_x \right) + c\tilde{\gamma} \tilde{V}_y \right) \\
 &= \frac{1}{\tilde{\rho} + \rho_d} \left(-\bar{V}_x |\tilde{\gamma}| \cdot |s\tilde{\gamma}| + \left(c\tilde{\gamma} \tilde{V}_y - s\tilde{\gamma} \tilde{V}_x \right) \tilde{\gamma} \right) \\
 &\leq \frac{1}{\rho_{\min}} \left(- (1 - \alpha) \left(\bar{V}_x - \tilde{V}_{x,\max} \right) |\tilde{\gamma}| \cdot |s\tilde{\gamma}| \right. \\
 &\quad \left. + \left(|c\tilde{\gamma}| \tilde{V}_{y,\max} - \alpha \left(\bar{V}_x - \tilde{V}_{x,\max} \right) |s\tilde{\gamma}| \right) |\tilde{\gamma}| \right)
 \end{aligned} \tag{8.45}$$

where $\alpha \in (0, 1)$. Let $\delta_1 = \arctan\left(\frac{\vartheta_\gamma}{\alpha}\right)$, $\Delta_1 = \pi - \arctan\left(\frac{\vartheta_\gamma}{\alpha}\right)$. Then,

$$\dot{V}_\gamma \leq -\frac{1}{\rho_{\min}} (1 - \alpha) \left(\bar{V}_x - \tilde{V}_{x,\max} \right) \sin(\delta_1) |\tilde{\gamma}| \quad \forall \tilde{\gamma} \in H(\delta_1, \Delta_1) \tag{8.46}$$

where $H(\delta_1, \Delta_1) = \{\tilde{\gamma} \in \mathbb{R} \mid \delta_1 \leq |\tilde{\gamma}| \leq \Delta_1\}$. Since $\boldsymbol{\xi}$ is the state of a system which is UPAS by assumption, by Theorem 7.2 the system (8.43) in cascade with the dynamics of $\boldsymbol{\xi}$ is UPAS on $\Theta_1 \times \Theta_2 \times \Theta_3$. $\square$

Proof of Corollary 8.1. By Proposition 8.2, the system (8.27) in cascade with (8.19) is UPAS on Θ_l . Then it follows from Proposition 8.3 that the complete cascaded system (8.19), (8.27) and (8.43) is UPAS on $\Theta_1 \times \Theta_2 \times \Theta_l$. $\square$

Remark. Since the nominal part of the dynamics (8.43b) is LES, there is no need (or possibility) of further diminishing the area to which $\tilde{\rho}$ converges in the nominal case. However, when the system (8.43b) is perturbed by another subsystem which is UPAS and therefore creates a generally non-vanishing disturbance, increasing k_ρ will help dominate the disturbance and decrease the area to which $\tilde{\rho}$ converges. Therefore, k_ρ is also included as a parameter when considering the practical stability of the complete system (8.43).

The domain on which the analysis in the proof of Proposition 8.3 is valid is illustrated in Figure 8.6.

Remark. The controllers (8.37), (8.41) can be applied to vehicles other than USRs, provided the vehicle is equipped with low-level controllers capable of (at least) practically asymptotically stabilizing the heading and velocity of the vehicle. The results of Proposition 8.3 can hold for states $\boldsymbol{\xi}$ consisting of other dynamic variables than those of the USR, so long as $\boldsymbol{\xi}$ contains $\tilde{v}_t, v_{n,\text{rel}}, \tilde{\theta}$ and that the dimensions of the interconnection terms $\mathbf{g}_1, \mathbf{g}_2$ in (8.43) are adjusted to match.

Remark. The parameter ϑ_γ is in practice not a freely adjustable control parameter, but a ratio of sizes of physical properties which influence the area to which the polar coordinate angle error $\tilde{\gamma}$ converges. The greater the smallest (i.e. worst-case) downstream current velocity $\bar{V}_x - \tilde{V}_{x,\max}$ is compared to the greatest sideways current $\tilde{V}_{y,\max}$, the smaller the area to which the polar angle error converges.

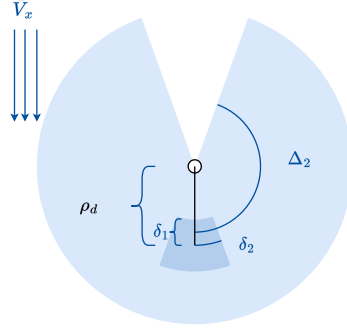


Figure 8.6: Illustration of the area in which the analysis of the system (8.42) is valid (light blue) and the area to which the position converges (dark blue)

8.4 Coupled solver for high-fidelity simulation

This section briefly summarizes the algorithm simulating a planar articulated swimmer in a complex fluid environment with fluid-structure interaction which is used to validate the proposed control design in Section 8.5. The method is presented in [13] and relies on a computational fluid dynamics (CFD) solver coupled with a multi-body system (MBS) solver.

For the CFD solver, a vortex particle-mesh (VPM) technique is used. The VPM method solves the incompressible flow past deforming geometries by using the velocity-vorticity formulation of the Navier-Stokes equations,

$$\frac{D\omega_f}{Dt} = (\omega_f \cdot \nabla)\mathbf{u}_f + \nu \nabla^2 \omega_f, \quad (8.47a)$$

where D/Dt denotes the Lagrangian derivative, $\mathbf{u}_f$ is the velocity field, ν is the kinematic viscosity and ω_f is the vorticity field. The method is summarized in Algorithm 1.

The first step in Algorithm 1 consists of recovering the velocity field from the vorticity field. Then, in the projection step, the fluid evolves as if the swimmer is absent. The resulting velocity field and position of particles, $\mathbf{u}_f^n$ and $\mathbf{x}$, are then used to predict the linear and angular momentum, $\mathbf{P}_{\text{proj}}$ and $\mathbf{I}_{\text{proj}}$, of the swimmer. The resulting forces and moments are given by $\mathbf{F}_{\text{proj}}$ and $\mathbf{M}_{\text{proj}}$, respectively. The hydrodynamical forces and moments are obtained through a mapping $\mathcal{F}$ and given by $\boldsymbol{\tau}_{\text{hyd}}$, while the actuation forces are given by $\boldsymbol{\tau}_{\text{act}}$. The MBS solver then computes the generalized coordinates given by $\mathbf{q} = [x_{\text{cm}}, y_{\text{cm}}, \theta_1, \phi_1, \dots, \phi_{N_l-1}]^T$, where N_l is the number of links, from the equations of motion. The configuration is then translated into a characteristic function that describes the swimmer's shape χ_s through the mapping $\mathcal{G}$. The velocity field of the structure is represented by $\mathbf{u}_s$, which is found in a similar fashion through the mapping $\mathcal{H}$. The no-slip condition is then enforced by use of Brinkman penalization in (8.52a), resulting in the new velocity and vorticity fields, $\mathbf{u}_\lambda$ and ω_λ respectively. The forces and moments resulting from constraining the fluid, $\mathbf{F}_{\text{pen}}$ and $\mathbf{M}_{\text{pen}}$, are calculated in (8.52c), and (8.52d). Finally, the vorticity field is updated in (8.53a). Additionally, the time-step is constrained so that $\Delta t^n \leq$

Algorithm 1 Coupled Solver

While: $t^n \leq t^{\text{End}}$

- 1: Retrieve velocity field from vorticity field by solving the Poisson equation

$$\nabla^2 \mathbf{u}_f^n = -\nabla \times \omega_f^n. \quad (8.48)$$

- 2: Calculate projection forces and moments

$$\mathbf{P}_{\text{proj}}^{n+1} = \int_{\Omega} \rho_f \chi_s^n \mathbf{u}_f^n d\mathbf{x}, \quad \mathbf{F}_{\text{proj}}^{n+1} = \frac{\mathbf{P}_{\text{proj}}^{n+1} - \mathbf{P}_{\text{proj}}^n}{\Delta t^n}, \quad (8.49a)$$

$$\mathbf{I}_{\text{proj}}^{n+1} = \int_{\Omega} \rho_f \chi_f^n (\mathbf{x} \times \mathbf{u}_f^n) d\mathbf{x}, \quad \mathbf{M}_{\text{proj}}^{n+1} = \frac{\mathbf{I}_{\text{proj}}^{n+1} - \mathbf{I}_{\text{proj}}^n}{\Delta t^n}. \quad (8.49b)$$

- 3: Time integration of MBS and update swimmer position and velocity

$$\boldsymbol{\tau}_{\text{hyd}}^{n+1} = \mathcal{F}(\mathbf{F}_{\text{proj}}^{n+1} + \mathbf{F}_{\text{pen}}^n, \mathbf{M}_{\text{proj}}^{n+1} + \mathbf{M}_{\text{pen}}^n), \quad (8.50a)$$

$$\boldsymbol{\tau}_{\text{act}}^{n+1} \rightarrow \text{provided by a control law}, \quad (8.50b)$$

MBS solver computes $\mathbf{q}^{n+1}$ and $\dot{\mathbf{q}}^{n+1}$ for $t^{n+1} = t^n + \Delta t^n$ with the forces calculated.

$$\chi_s^{n+1} = \mathcal{G}(\mathbf{q}^{n+1}), \quad \mathbf{u}_s^{n+1} = \mathcal{H}(\mathbf{q}^{n+1}, \dot{\mathbf{q}}^{n+1}). \quad (8.51a)$$

- 4: Penalization of vorticity field and calculation of penalization forces and moments

$$\mathbf{u}_{\lambda}^{n+1} = \frac{\mathbf{u}^n + \lambda_f \Delta t^n \chi_s^{n+1} \mathbf{u}_s^{n+1}}{1 + \lambda_f \Delta t^n \chi_s^{n+1}}, \quad (8.52a)$$

$$\omega_{\lambda} = \nabla \times \mathbf{u}_{\lambda}^{n+1}, \quad (8.52b)$$

$$\mathbf{F}_{\text{pen}}^{n+1} = \int_{\Omega} \lambda_f \rho_f \chi_s^{n+1} (\mathbf{u}_{\lambda}^{n+1} - \mathbf{u}_s^{n+1}) d\mathbf{x}, \quad (8.52c)$$

$$\mathbf{M}_{\text{pen}}^{n+1} = \int_{\Omega} \lambda_f \rho_f \chi_s^{n+1} \mathbf{x} \times (\mathbf{u}_{\lambda}^{n+1} - \mathbf{u}_s^{n+1}) d\mathbf{x}. \quad (8.52d)$$

- 5: Time integration of vorticity field

$$\frac{\partial \omega_{\lambda}}{\partial t} = \nu \nabla^2 \omega_{\lambda} - \nabla \cdot (\mathbf{u}_{\lambda} \omega_{\lambda}) \quad (8.53a)$$

$$\omega_f^{n+1} = \omega_{\lambda}^{n+1} \quad (8.53b)$$

End while.

$\min\{C, h^2/2\nu, \Delta t^{\max}\}$ where $\Delta t^n = (t^{n+1} - t^n)$, C are the Lagrangian Courant-Friedrich-Levy conditions (LCFL) [13], and h is the uniform spacing in the Cartesian discretization grid. The LCFL conditions may allow for time-steps that destabilize the MBS solver, therefore the time-step is constrained by a maximal time-step $\Delta t^{\max}$ which was found empirically.

While the CFD solver and the over-all architecture of the scheme is implemented as described by [13], the equations describing the swimmer dynamics in [13] are substituted with the USR dynamics formulation found in [74, 76]. The terms in the model from [74, 76] associated with hydrodynamical effects are then replaced with the forces and moments computed in the CFD solver. For more details regarding the incorporation of the USR dynamics see [117].

8.5 Simulation results

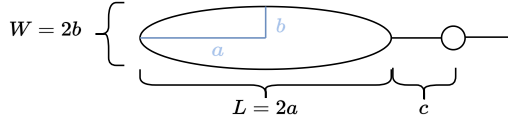
This section presents results from a simulation study which was conducted in order to validate the performance of the proposed controller. The study is performed utilizing the high-fidelity simulation method described in Section 8.4, which faithfully captures the evolution of the fluid as well as the interaction between the fluid and the USR, in contrast to the simplified models which the control design and analysis are based on. For this reason, the controller's performance can be demonstrated with greater confidence when encountering complex, realistic fluid effects which it was not explicitly designed to handle. Moreover, one can observe how properties derived in the stability analysis carry over to the high-fidelity simulations.

8.5.1 Simulation setup

This section describes the parameters used during the simulations. The spatial domain is $[0, 3.0] \times [0, 3.0]$ with a discretization grid resolution of [512, 512]. For the velocity reference (8.41), $k_\rho = 0.2$ was chosen. The remaining parameters are shown in Table 8.1. The swimmer dimensions are illustrated in Figure 8.7. A circular rigid body is placed in the center of the spatial domain, and a current is applied to the fluid, resulting in a vortex street forming downstream from the bluff body. The centroid of the bluff body is specified by $[x_{\text{cyl}}^{\text{cm}}, y_{\text{cyl}}^{\text{cm}}]^\top$. The Reynolds number chosen is $\text{Re} = \frac{\bar{V}_x D_{\text{cyl}}}{\nu} = 100$ for all simulations, where D_{cyl} is the diameter of the circular rigid body and the kinematic viscosity is given by ν . The Reynolds number for the cylinder is chosen rather than that of the particular swimmer to ensure consistent wake characteristics. It is desirable to have a standardized configuration of vortex shedding around the cylinder since it allows for the investigation of various swimmer sizes and control methods while maintaining a uniform environment. For a USR of a similar size to Mamba [95], the resulting kinematic viscosity becomes significantly higher than that of seawater. This choice was made for reasons of computational efficiency. However, the qualitative difference only becomes salient in the 3D case, where shedding occurs in cells along the height of the cylinder for $\text{Re} \in [300, 3 \cdot 10^5]$ [147]. Thus, the simulation with increased viscosity still captures the qualitative behaviour of the vortex street. The dimensionless time is defined as

Table 8.1: Simulation parameters

D_{cyl}	a	b	c	ρ_f	ρ_s	C	Δt^{max}	$x_{\text{cyl}}^{\text{cm}}$	$y_{\text{cyl}}^{\text{cm}}$
0.1	$0.3125D_{\text{cyl}}$	$0.2a$	$0.1D_{\text{cyl}}$	997	997	0.02	0.001	1.5	1.5
$\bar{V}_x$	k_ρ	k_i							
$2.0D_{\text{cyl}}/s^*$	0.2	0.01							


Figure 8.7: Link and joint configuration

$t^* = \frac{tU}{D_{\text{cyl}}}$ with the dimensionless unit s^* . A snapshot of the simulation of the USR in the wake can be seen in Figure 8.8.

The control input (8.18) requires exact knowledge of the hydrodynamical forces and moments that affect the USR when it operates in the wake. Measurements of these terms are not available in the high-fidelity simulation. This would also be the case in a real-life scenario, as there are no direct measurements of the forces and moments applied from the wake on the USR. Therefore, cancellation of hydrodynamical forces and moments is not implemented in the controller for this simulation study.

During initial tests for the simulation study, large deviations from the desired tangential velocity and desired distance to the bluff body were observed. This may be explained by unmodeled dynamics and disturbances that occur in the Von Karman vortex street. To combat this, integral action was added to the velocity reference generated by the high-level controller (8.41), using an integral gain $k_i = 0.01$. This addition and its relation to the theoretical analysis presented in Section 8.3 is further discussed in Section 8.5.2.

The initial tangential velocity of the USR is set to 0. This results in a large initial error between the desired tangential velocity and the initial tangential velocity, which demands a large virtual input $\ddot{\lambda}$. This can be interpreted as the rate of change in the frequency of the gait encoded by the joint references. Fast motion of the USR induces large vorticity spikes that propagate through the fluid environment, increasing the computational cost of the simulation. To avoid the extended computational time and disturbances in the vortex wake, a saturation, $\ddot{\lambda}_{\text{max}} = 7 \text{ rad/s}^2$, is imposed on $\ddot{\lambda}$. In a real-world scenario, this would likely only be part of an initial deployment procedure and only happen at the start of a mission.

Four simulation cases were investigated. Firstly, the USR was simulated with its initial position outside of the wake, with a static current direction. Secondly, the USR was simulated with its initial position in the wake, with the current direction rotating slowly. In the third case, the USR was simulated with its initial position in the wake, with the current direction rotating more quickly. Finally, the previous cases were combined in the fourth case, where the initial position of the USR was

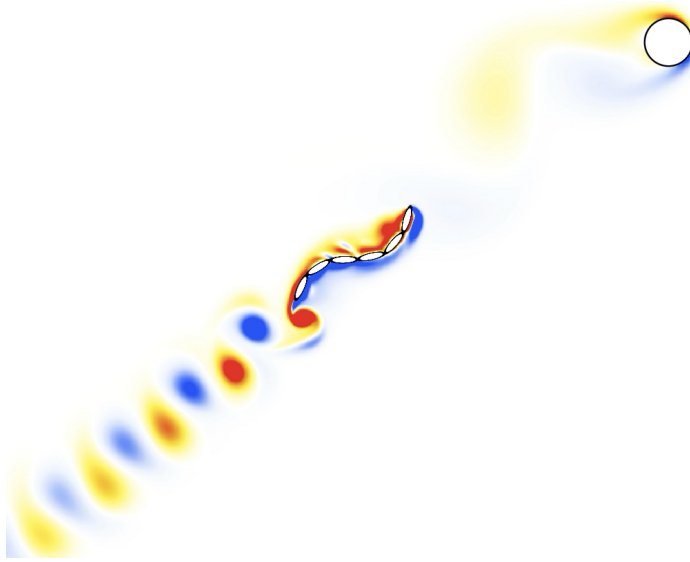


Figure 8.8: Snapshot from the simulation of the USR in a vortex street

placed outside of the wake, and the direction of the current rotating. The rotation of the current is faster than in the second case but slower than in the third case.

8.5.2 Convergence to a wake with static current direction

The first scenario, where the USR starts outside of the wake, investigates the ability of the control law to stabilize the USR to the wake from relatively far away. This would correspond to the USR entering a wake initially, in order to initiate charging. This scenario is simulated both with and without adding integral effect to the forward velocity controller. Figure 8.9 shows the results of the simulation case without any integral effect. Figures 8.9a to 8.9c demonstrate that the low-level control objectives of Section 8.2 are achieved. Specifically, Figure 8.9a shows how the joint angles track their reference quite well. One may observe a small discrepancy between the joint angles and reference, which is in fact not strictly in keeping with Proposition 8.1. This may be explained by the complex hydrodynamical forces which were assumed to be cancelled in the analysis. In the simulation these forces are not available for feedback, as would also be the case in a real-life scenario, and they provide a disturbance. However, the GES property of the nominal control system guarantees a certain robustness in the presence of disturbances [78, Lemma 9.1], resulting in the overall successful tracking of the joint references seen in Figure 8.9a. Moreover, Figures 8.9b and 8.9c show how the heading and tangential velocity converge to a neighbourhood of their respective references, in accordance with Proposition 8.2. In Figure 8.9a, only the first joint angle error is plotted for greater clarity, as the tracking in ϕ is similar for the remaining joint angles. Figures 8.9d and 8.9e show the regulation of the high-level references for ρ and γ designed in Section 8.3. It is seen that γ converges to a neighbourhood around β_c , and that ρ converges to a neighbourhood around ρ_d , although a clear stationary deviation from the reference is

observed in ρ . This is to be expected, as the velocity reference attempts to counteract the unmodeled disturbance from the current without any integral action. Still, the high-level control objectives are achieved in accordance with Proposition 8.3, as ρ converges within some neighbourhood of ρ_d . In the case of a steady-state offset, the result of Proposition 8.3 formally states that a greater proportional gain will reduce the steady-state offset but not necessarily remove it entirely. This motivates the addition of an integral term to the control law (8.41) in the next simulation.

In Figure 8.10, the performance of the controller with integral action in forward velocity is shown in the same scenario. For the low-level control objectives in Figures 8.10a to 8.10c as well as for γ in Figure 8.10d, the performance is comparable with that of the controller without integral action seen in Figure 8.9. However, in Figure 8.10e it is shown that ρ converges to a significantly smaller neighbourhood of ρ_d than is seen in Figure 8.9e. While a stationary error is observed in the case without integral action, in Figure 8.10e it can instead be seen that after an initial reaching phase, a steady-state behaviour is achieved where the radius oscillates around the reference. Both this and the offset present without the integral action are within the realm of expected behaviour of a UPAS system, since they are both examples of the state converging to and remaining within a neighbourhood of the reference. However, the integral action has not been included in the analysis in Section 8.3. Integral action is typically used to combat constant disturbances, in which case an integral state can often be shown to converge to a value exactly offsetting the disturbance. In the presence of a time-varying disturbance, and when the other states are practically stable rather than converging to an exact value, it is difficult to determine what value an integral state would converge towards and subsequently what stability properties the resulting system might have. However, it can be argued that even if the radius is UPAS regardless of integral action, oscillation around the reference is a more desirable steady-state behaviour than oscillation around an offset point for the given application. While the error could also be reduced by increasing the proportional gain instead of adding an integral term, a very large proportional gain can result in an excessive and unrealistic velocity reference in response to large initial errors. Thus, the integral action improves the controller performance significantly in practice in the given scenario.

To summarize, the success of the cascaded controller in stabilizing both γ and ρ to a vicinity of their references demonstrates the ability of the controller to automatically align the USR with the wake without knowledge of the ocean current's direction or magnitude. The control laws as implemented in the simulation do not rely on cancellation of hydrodynamical terms which are difficult to measure or estimate in practice, making these results promising with regard to a potential future practical implementation.

8.5.3 Tracking of a wake in slowly rotating current direction

This section presents the results from the simulations with the USR starting within the wake, with the current direction rotating slowly. This scenario investigates the ability of the control law to track changes in the wake that occur if the direction of the greater ocean current varies, even though the control law was developed under the premise of a constant current direction. The current direction is kept constant

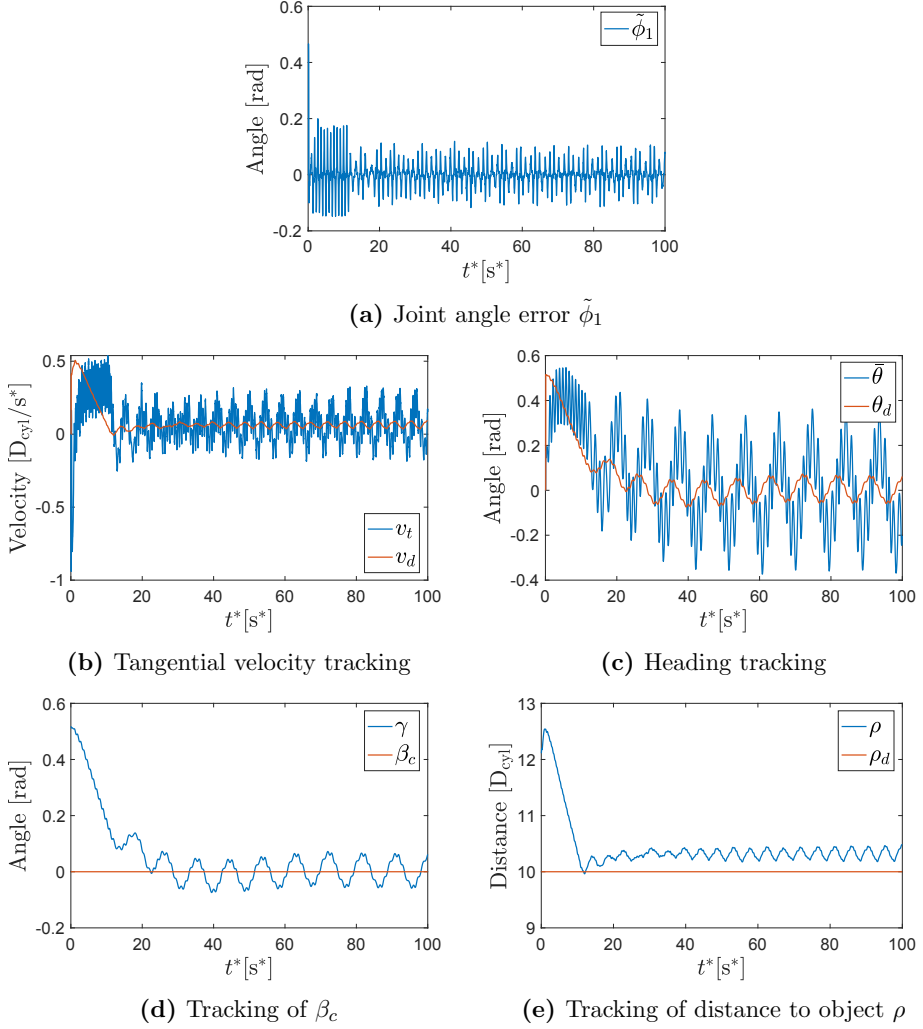


Figure 8.9: Plot of tracking for the low-level and high-level controllers without integral action in the case of regulation from outside a non-rotating wake

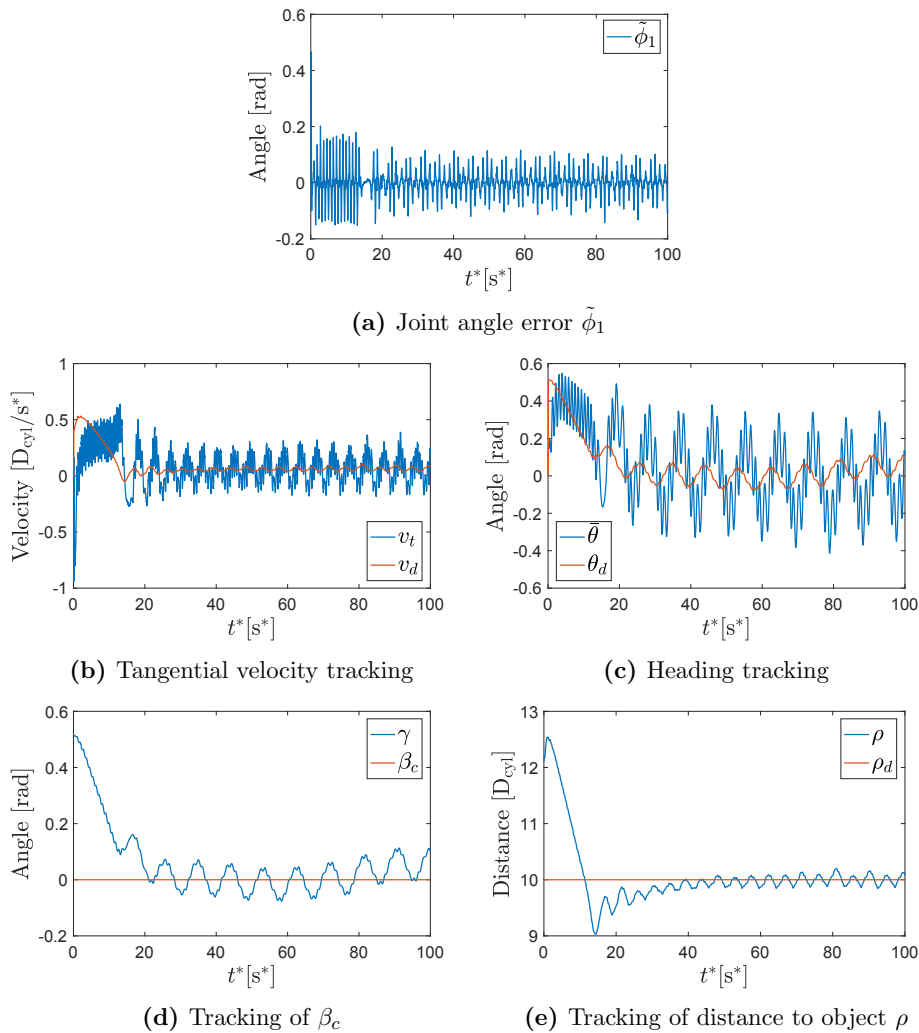


Figure 8.10: Plot of tracking for the low-level and high-level controllers with integral action in the case of regulation from outside a non-rotating wake

until $t^* = 10$ in order to allow the formation of a vortex street prior to rotating the direction. Figures 8.11a to 8.11c demonstrate that the low-level control objectives of Section 8.2 are achieved. Specifically, Figure 8.11a demonstrates that the joint angles converge to their references in accordance with Proposition 8.1, although small discrepancies can be observed. Again, as in the case of a static wake, these discrepancies are explained by the unmeasured hydrodynamical effects which are not accounted for during analysis. For clarity, again only the first joint angle error $\hat{\phi}_1$ is plotted, as the remaining joint angle errors exhibit similar behaviour. Moreover, Figures 8.11b and 8.11c show how the heading and tangential velocities converge to a neighbourhood of their respective references, in accordance with Proposition 8.2. Furthermore, the heading is shown to track the moving reference quite closely during the rotating ocean current.

Figures 8.11d and 8.11e show the tracking of the high-level control objectives from Section 8.3. Specifically, ρ is shown to converge to a neighbourhood around its reference, although some greater fluctuation is seen here than in Figure 8.10e. Moreover, while the current now rotates at non-zero rate, unlike what the error dynamics (8.39) are based on, γ still tracks its moving reference β_c quite well, although some phase lag is observed. This is to be expected, as the controller is designed for a stationary direction, and has to reactively correct for changes without any knowledge of the motion. In addition, the USR has no actuation in its normal direction, in which it has to move to track the wake. Instead, it is pushed by the ocean current, which has little to no contribution along the tangent of the circle when the USR is aligned with the wake. Thus a deviation from the centerline must occur in order for a restoring force to appear. For the considered energy-harvesting application this phase lag is acceptable as long as the USR ends up stabilizing itself in the middle of the street when the current is no longer rotating. The change in direction of an ocean current is expected to be far slower in a real-world application than in the simulation scenario presented here. The controller achieves the control objective even in this exaggerated scenario. This is promising with regard to a real-life scenario.

8.5.4 Tracking of a wake in current with quicker change of direction

This section presents the results from the simulations with the USR starting within the wake, with the current direction rotating more quickly. This scenario is included to investigate the limits of the ability of the control law to track changes in the vortex wake under conditions which are far from the premises of the theoretical analysis. The current direction is kept constant until $t^* = 10$ in order to allow the formation of a vortex street prior to the direction rotating. Figure 8.12 show the results from the simulation case. The simulation is run for an entire revolution of the current. Figure 8.12a shows that the joint controller tracks the reference well, in accordance with Proposition 8.1. As in previous cases, there are small discrepancies which, while contradicting the claim of Proposition 8.1 that the origin is GES, can be explained by unmodeled hydrodynamical forces. Figures 8.12b and 8.12c demonstrate that the velocity and heading controllers to a large extent, achieve convergence to neighbourhoods of their references, with the exception of the velocity controller during the first 20 s of the simulation. The failure to achieve convergence

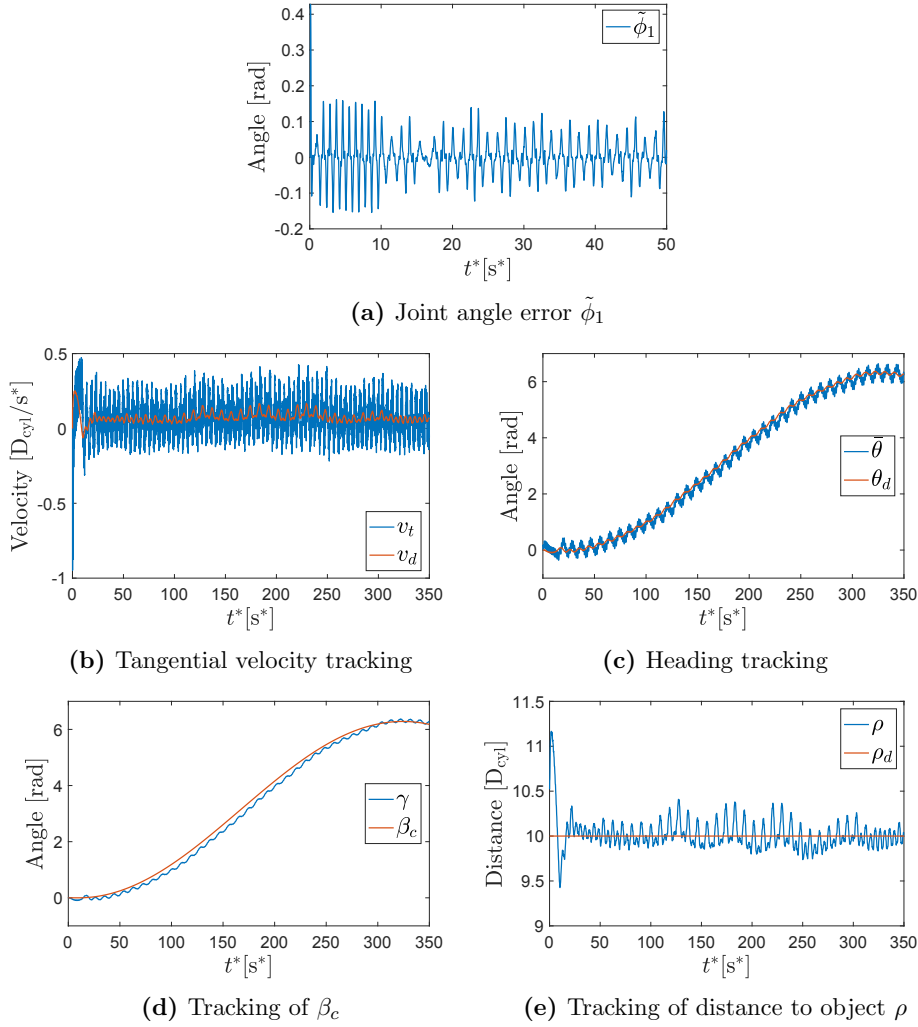


Figure 8.11: Plot of tracking for the low-level and high-level controllers in the case of a slowly rotating wake

to the reference during this period is caused by the saturation imposed on $\ddot{\lambda}$ during simulations. However, after the first 20 s*, the tangential velocity converges to a neighbourhood of the desired tangential velocity.

Figures 8.12d and 8.12e demonstrate that the controller still retains the ability to track the position references to a certain extent, although performance is degraded when compared to the previous observed cases. In Figure 8.12d it is shown that γ still remains within a neighbourhood following the reference value, although with a more pronounced phase lag. However, Figure 8.12e shows that while ρ at times converges to a neighbourhood of the reference, it also drifts a little away at times. By the end of the simulated time, the current again comes from its initial direction. The performance is not expected to improve nor reach a steady-state if simulated for a longer time over several revolutions of the current, since this scenario is outside of the expected operating conditions for the developed controller. Looking again at Figure 8.12e, it is of particular interest that the drift away from the desired radius occurring at around 55-70 s* coincides with the largest experienced heading errors, as can be seen in Figure 8.12c. The heading error disturbs the position dynamics due to the interconnection of the control layers. In addition, the velocity reference (8.41) is scaled down when there is a heading error, i.e. when the USR does not point directly towards the bluff body. While this can help avoid the unnecessary control of swimming in the wrong direction, it means that performance of the control of radius is reduced until the desired heading is reached again.

It should be noted here that this case significantly violates the premise that the direction of the current is constant, on which the error dynamics and hence stability analysis of γ are based. As such some degradation of performance and loss of stability properties are to be expected. Taking this into consideration, the controller seems to retain the ability to achieve the control objectives fairly well. In addition, if the current changes its direction too fast for the USR to follow, the USR risks ending up in a situation where it needs to swim with the current, violating Assumption 8.2. If the USR is required to swim with the current, the current itself might be fast enough to meet the total reference velocity, in which case the controller would cause the USR to slow down or even stop swimming and drift with the current. If the USR stops actively swimming, it also loses the ability to turn, similarly to a ship with a rudder. This further hinders the USR from recovering.

Overall, both the low-level and high-level controllers track their references quite well, demonstrating the UPAS properties of the system as a whole. Given the high-fidelity simulations which were carried out to verify the controller performance, and given the low reliance of the controller on model knowledge, these simulations appear greatly promising for the method's efficacy in experiments and transfer to real-world use cases.

8.5.5 Tracking of a wake with rotating current direction and initial position outside of wake

This section presents the results from the simulations with the USR starting outside of the wake and a rotating current direction. The rotational velocity of the current is selected to be faster than the second case but slower than the third case. This is done to investigate the behaviour of the USR with the suggested controller when all

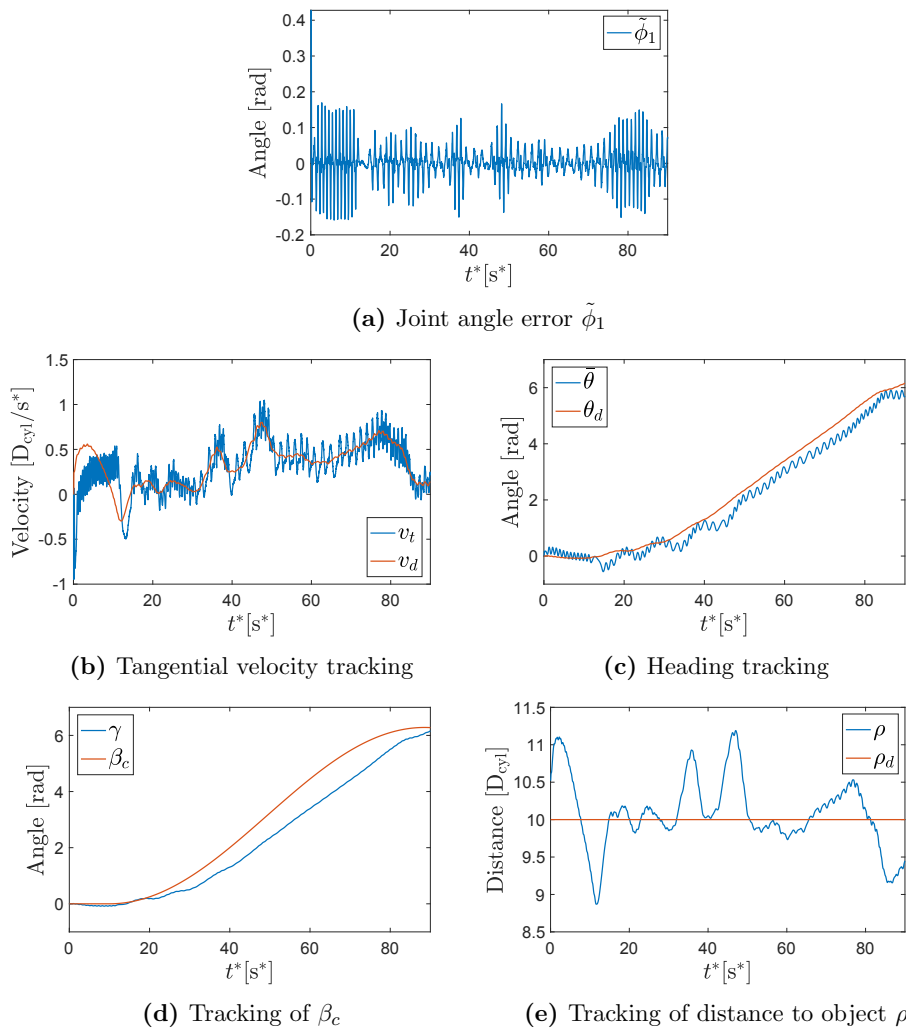


Figure 8.12: Plot of tracking for the low-level and high-level controllers in the case of a quickly rotating wake

the conditions of the previous cases are combined. The direction of the current is kept constant until $t^* = 4.0$ to allow for the formation of the vortex wake. The time before the current direction rotates is shortened in order for the current to change direction before the USR reaches the vortex wake.

The simulation results are shown in Figure 8.13. Figure 8.13a shows the joint tracking error for the first joint, and as in the previous cases the joint controller is shown to track the reference well. Interestingly, the tracking error decreases at around the same time as the tangential velocity and its reference, shown in Figure 8.13b, decrease from their initial, higher values. In other words the joint tracking improves as the reference gait stabilizes to a slower frequency. The gait frequency slows down after the USR achieves the desired position and starts to maintain a steady position against the current. Figures 8.13c and 8.13d show that both the heading and γ have an initial converging phase after which they track their desired values reasonably well, but with a slight delay. This is to be expected, especially in the case of γ , since the controller is designed for a stationary direction and will therefore exhibit some delay in tracking a moving direction. Compared to the two previous cases with a rotating wake, γ is shown to lag further behind β_c now, in Figure 8.13d, than in Figure 8.11d, but not as much as in Figure 8.12d, which is as expected when the wake now rotates at a speed between the two previous cases. Figure 8.13d also shows that as the rotation slows down towards the end of the simulated time, γ converges closer to β_c . Figure 8.13e shows that after an initial overshoot, the radius converges to a neighbourhood of its desired value. The overshoot is larger than in the case of converging to a stationary wake in Figure 8.10e, but it is worth noting that the initial state in Figure 8.13e is also further away from the reference than the one in Figure 8.10e.

Overall, the controller manages its objective of reaching and tracking the wake well also in the case of having to converge towards a rotating wake. It is worth repeating that both the speed at which the current rotates, as well as the starting time of the rotation had to be made faster compared to the case in Section 8.5.3, in order for the wake to start moving away from the USR before the USR has converged to the vicinity of the wake. Taken together with the fact that the current rotation was unrealistically fast already in the previous scenario in Section 8.5.3, this demonstrates that the system dynamics with the proposed controller are sufficiently fast compared to the environmental effects it is designed to adjust to.

8.5.6 Investigation of the impact of parameters on the radius of convergence

The UPAS property which has been derived for the closed-loop system in Section 8.3 implies that the area of convergence for the state can be altered by adjusting the parameters ϑ_l , k_ρ , ϑ_γ . It is important to verify that these properties, which are proven with simplified dynamical models, still hold for the closed-loop system with a high-fidelity description of the system dynamics. To this end, a simulation study was conducted, investigating the effect of the parameter k_ρ on the radius of convergence of (8.43b). The USR is initially in a vortex wake with a static direction, further away from the cylinder than ρ_d . The simulation is performed with different proportional

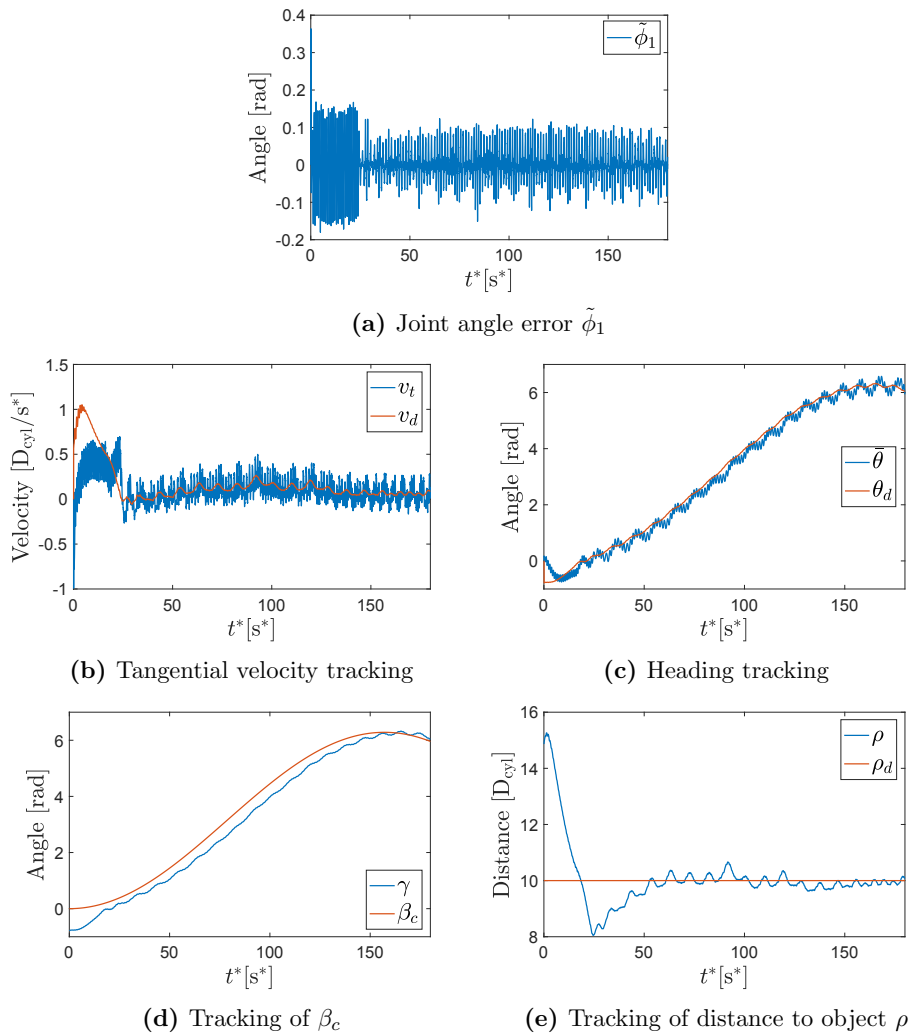


Figure 8.13: Plot of tracking for the low-level and high-level controllers in the case of a rotating current and USR with initial conditions outside of the wake

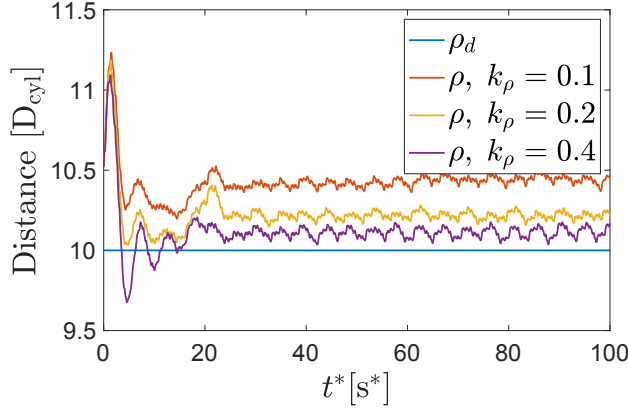


Figure 8.14: Tracking of distance to object ρ for different values of k_ρ . For the case with integral effect, $k_i = 0.05$

gains $k_\rho \in [0.1, 0.2, 0.4]$. Figure 8.14 shows the resulting plots of ρ for different values of k_ρ . The results show that the simulations with pure proportional action converge to neighbourhoods around ρ_d whose radii decrease with increasing values of k_ρ , which is in accordance with the previously stated property of UPAS systems. This demonstrates that the model and method used for analysis are sufficient to capture properties of the more complex model and the high-fidelity simulation described in Section 8.4. This is again promising for the control system's performance in real-world experiments and applications, as the high-fidelity simulation captures complex fluid effects that are not explicitly accounted for during analysis.

8.6 Chapter summary

In this chapter, the automatic alignment of a USR with the wake behind a bluff body was investigated. A cascaded controller was derived and the resulting closed-loop system was proven to be UPAS. Furthermore, the performance of the controller was verified through high-fidelity simulations of the USR-fluid system. The simulations successfully verified the theoretical properties of the controller, and demonstrated the efficacy of the controller both in stabilizing the USR to a vortex wake from outside of it, as well as in tracking changes in the wake direction due to rotations in the larger current direction.

The high fidelity of the simulations carried out, as well as the low requirements of model knowledge in the development of the controller, make these results promising with respect to the controller's performance in experiments and real-world scenarios.

Chapter 9

Conclusions and Future Work

This thesis has presented control methods for several potential uses of AIAUVs and USRs. Part I dealt with topics on the control of AIAUVs. Therein, Chapters 3 to 5 considered control methods for robot manipulators and VMSs, in particular AIAUVs, performing multiple tasks. Strictly prioritized tasks with the priority being enforced through use of null-space projections were considered in Chapters 3 and 4, while in Chapter 5 optimization of a common cost function was considered, in which tasks can be assigned soft priority through weighting. Control of AIAUVs as a vehicle was considered in Chapter 6. Part II considered control of swimming USRs, and related theory on uniform practical asymptotic stability relevant for the analysis of the proposed control method. A new theorem for cascaded systems consisting of UPAS subsystems was presented in Chapter 7. In Chapter 8, control of swimming USRs was considered, specifically for the case of automatically aligning the USR with a wake behind a bluff body.

The first topic, treated in Chapters 3 and 4, was that of control methods for accomplishing multiple prioritized tasks. The control methods considered in these chapters are based on the use of null-space projections to enforce priority by ensuring that lower-priority tasks do not interfere with higher-priority ones. The system dynamics are transformed into dynamically consistent task coordinates, so that the dynamics of the tasks are inertially decoupled. In Chapter 3, a new robust tracking control law was proposed and shown to give exponential stability of all task tracking errors, and global exponential stability in the absence of any singularities. Then, new analysis of existing tracking control laws was presented, proving stronger stability properties than in previous literature. The performance of the newly proposed control law was investigated and compared against similar existing control laws in simulations, which demonstrated that the proposed control law performs similarly to a first-order sliding mode control law, and suggest that the choice to use original task velocities in the damping term may improve the convergence of lower-priority tasks. The proposed control law is still prone to chattering in the control input, similarly to first-order sliding mode control. In order to achieve a continuous control input while achieving robustness against disturbances and unmodeled effects, higher-order sliding mode methods could be applied within the framework of dynamically consistent task coordinates. In this case, formulating the control law entirely in dynamically consistent velocities lends itself well to sliding mode control

due to the resulting cascaded structure of task dynamics and kinematics. Whether higher-order sliding mode techniques could be combined with some terms expressed in original task velocities as in the control law proposed in this thesis, with the aim of achieving better transient performance of lower priority tasks, requires further investigation. In addition, the performance of the proposed control method and similar method should be further compared and validated through additional simulations involving physical interaction scenarios, as well as experiments with both fixed-base manipulators and VMSs.

In Chapter 4, dynamically consistent task coordinates were used for control the CG of free-floating VMSs as a means of repositioning the system, with the aim of reducing the need to counteract disturbances to the position caused by joint motion. The GJM, originally developed to control the joints of VMSs in space without inducing total momentum and thus drift to the overall system, was shown to be a special case of the transformation to dynamically task consistent coordinates. This insight was leveraged to extend previous control methods developed for free-floating VMSs in space to tracking of time-varying trajectories for an arbitrary number and choice of lower-priority tasks to be performed by the VMS. A control allocation scheme for AIAUVs was proposed to ensure that thrusters will be engaged only by the highest priority task of positioning the overall system, while the remaining tasks are performed using the joints. The proposed control method is demonstrated to perform well in a simulation study, even when added mass effects are not taken into account and compensated for by the controller. Although reducing the use of thrusters and consequently reducing the energy consumption served as motivation for choosing to control the CG of the AIAUV, this approach has yet be compared against previous control methods. In the future, more simulations should be performed in order to compare the performance and expended control effort of the proposed control method against other choices of tasks. Another interesting direction of research could be to avoid cancelling out hydrostatic momentum, but instead take into account the change of the hydrostatic equilibrium depending on the joint configuration, and then control only the rotation about the axis along the direction of gravity. Finally, the proposed control method would have to be made robust, for example applying the control laws considered in Chapter 3, to make it applicable in a real-life scenario with uncertain hydrodynamic effects.

In contrast to using null-space projections to enforce strict priority of multiple control objectives, in Chapter 5 all control objectives were gathered in a common cost function in a mixed H_2/H_∞ control problem to combine optimal control with robustness. The mixed H_2/H_∞ control problem was formulated as a differential game. In order to construct an approximate solution to this problem, feedback linearization was applied to the joints of an AIAUV to simplify their dynamics into double integrator dynamics. An algebraic $\bar{P}$ matrix solution, necessary to construct an approximate optimal solution to the mixed H_2/H_∞ control problem, was found for the double integrator dynamics. This solution was then applied to control the joints of the AIAUV to desired setpoints while avoiding self-collisions and joint limits. Simulation results demonstrate that collisions are successfully avoided while regulating the joints to desired setpoints, also in the case of parameter uncertainties in the feedback linearizing controller. The approximate optimal solution, i.e. control input,

relies on a dynamic extension which must be initialized such that the future system trajectories remain within a region of the state space. Finding such a suitable initial value for the dynamic extension can be very challenging. To address this, allowing the value of the dynamic extension to jump in order to satisfy the necessary inequalities using the method from [110] should be considered in the future. In addition, in this thesis, the control method was applied directly to the joints of an AIAUV, which would require a planning module to generate setpoints required to perform more meaningful tasks. In order for the method to be more expressive and suitable for VMSs, applying it to task coordinates rather than original system coordinates could be considered.

Control of an AIAUV when using it as a vehicle was considered in Chapter 6, where the joints of an AIAUV were used for direction control to achieve path following in 3D. First, a control law for tracking a time-varying desired pointing direction, or reduced attitude, was developed for a general rigid body equipped with two control torques. Based on this control law, references for the joint angles of an AIAUV were chosen in order to point the AIAUV in the direction given by a guidance law, while using thrusters for forward propulsion. The proposed control scheme was demonstrated to perform well in simulations. However, it performs better for vehicles which are not ballasted so that they are passively stabilized in roll. To further improve this control method, the hydrostatic restoring moment acting on the vehicle as a whole should be compensated for, either by modifying the joint references accordingly, or by using additional thrusters to improve performance at the cost of increased thruster use. Furthermore, deriving and studying the turning dynamics of a curved AIAUV would enable the design of a control law which tracks the desired pointing direction more precisely, and the stability properties of which could be formally analysed. The reduced attitude tracking control law could also be extended to the dynamics of traditionally shaped AUVs and applied to e.g. path following or formation control for underactuated underwater vehicles. Finally, it would be interesting to test the proposed control method in full-scale experiments.

In Part II, Chapter 7 presented a new theorem for proving UPAS of cascaded systems consisting of UPAS subsystems. The theorem was applied in Chapter 8, where a control method for automatic alignment of swimming USRs with the wake behind a bluff body was developed. Controllers for the heading and forward velocity were designed, and combined with WOHC to point the USR towards the bluff body while maintaining a desired distance to it. It was then shown that the position of the USR converges towards the desired position behind the bluff body as a consequence of the ocean current pushing the USR downstream. The performance of the controller was verified through high-fidelity simulations of the USR and surrounding fluid. The simulation results demonstrated that the control method performs well, and is also able to track a slowly rotating wake. In the simulation study, an additional integral term not present in the theoretical analysis was added in order to avoid an offset in the desired distance to the bluff body. In the future, an analysis of the resulting closed-loop system with the integrator term should be performed. Since the control objectives are practically asymptotically stable and therefore do not converge exactly, it is not reasonable to expect that the integrator state will converge to a fixed value. This will complicate a future stability analysis, and may require new

combinations or extensions of existing stability theory.

While energy efficiency or even autonomy has been a motivating factor in the development of several of the control methods in this thesis, much investigation and development still remains to validate their benefits, and before they can be applied in practice. For the control methods presented in Chapters 4 and 6 which were motivated by reducing the energy consumption, it remains to compare the energy expended by these methods against other existing approaches. Simulation studies may provide some estimates of the work performed by the thrusters and joints. The true energy consumption, however, depends on the energy consumed by the individual actuators, and would need to be investigated experimentally. In the case of energy autonomy and energy harvesting, which served as motivation for development of the control scheme in Chapter 8, much work still remains. This includes development of both control methods enabling a USR to transition from active station-keeping to passively harvest energy, as well as hardware with the necessary recharging capabilities.

Appendix A

Notation and Stability Definitions

Notation

The set of all non-negative real numbers is denoted $\mathbb{R}_{\geq 0}$. The two- and three-spheres are denoted $\mathbb{S}^2$, and $\mathbb{S}^3$, respectively. The class of continuously differentiable functions, i.e. functions such that the function and its first derivative are continuous, is denoted C^1 .

Vectors and matrices

The $n \times n$ identity matrix is denoted $\mathbf{I}_n$. An $n \times m$ zero matrix is denoted $\mathbf{0}_{n \times m}$. If dimensions are apparent from the context, $\mathbf{0}$ may be used to denote a zero vector or matrix.

A diagonal matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ with entries $a_1, \dots, a_n$ on its diagonal is denoted $\mathbf{A} = \text{diag}(a_1, \dots, a_n)$.

For a vector $\mathbf{x} = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$, the matrix $\mathbf{S}(\mathbf{x})$ is defined as

$$\mathbf{S}(\mathbf{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \quad (\text{A.1})$$

and $\mathbf{S}(\mathbf{x})\mathbf{y} = \mathbf{x} \times \mathbf{y}$ for $\mathbf{y} \in \mathbb{R}^3$.

The smallest and largest eigenvalues of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are denoted $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$, respectively. Analogously, the largest singular value of a matrix $\mathbf{B} \in \mathbb{R}^{n \times m}$ is denoted $\sigma_{\max}(\mathbf{B})$.

Sets

The interior of a set $\mathcal{A}$ is denoted $\text{int}(\mathcal{A})$. The distance from some $\mathbf{x} \in \mathbb{R}^n$ to a set $\mathcal{A} \subset \mathbb{R}^n$ is

$$\|\mathbf{x}\|_{\mathcal{A}} = \inf_{\mathbf{z} \in \mathcal{A}} \|\mathbf{x} - \mathbf{z}\| \quad (\text{A.2})$$

where $\|\cdot\|$ is the Euclidean norm.

A ball of radius δ around a point $\mathbf{x} \in \mathbb{R}^n$ is denoted $\mathcal{B}_\delta(\mathbf{x}) = \{\mathbf{z} \in \mathbb{R}^n \mid \|\mathbf{z} - \mathbf{x}\| < \delta\}$. For a closed set $\mathcal{A} \subset \mathbb{R}^n$, $\mathcal{B}_\delta(\mathcal{A}) = \{\mathbf{z} \in \mathbb{R}^n \mid \|\mathbf{z}\|_{\mathcal{A}} < \delta\}$. The simplified notation $\mathcal{B}_\delta = \mathcal{B}_\delta(\mathbf{0})$ is used for a ball centered at the origin, and a closed ball of radius δ centered at the origin, i.e. the closure of $\mathcal{B}_\delta$, is denoted $\bar{\mathcal{B}}_\delta$. To further simplify notation, let $\|\mathbf{x}\|_\delta = \|\mathbf{x}\|_{\bar{\mathcal{B}}_\delta}$.

Class $\mathcal{K}$ and $\mathcal{KL}$

Definition A.1 (Class $\mathcal{K}$, [78], Definition 4.2). A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Definition A.2 (Class $\mathcal{KL}$, [78], Definition 4.3). A continuous function $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s , the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r , the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

Stability definitions

Consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \quad (\text{A.3})$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f} : [0, \infty) \times D \rightarrow \mathbb{R}^n$ where $D \subset \mathbb{R}^n$ is a domain containing the origin. The solution of (A.3) with initial state $\mathbf{x}_0$ and initial time t_0 , at time t , will be denoted $\mathbf{x}(t, t_0, \mathbf{x}_0)$.

Autonomous systems

Consider an autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (\text{A.4})$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f} : D \rightarrow \mathbb{R}^n$ is defined on a domain $D \subset \mathbb{R}^n$ and satisfies $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, such that the origin is an equilibrium point of the system (A.4).

Definition A.3 ([78], Definition 4.1). The equilibrium point $\mathbf{x} = \mathbf{0}$ of (A.4) is

- stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|\mathbf{x}(0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| < \varepsilon, \quad \forall t \geq 0,$$

- asymptotically stable if it is stable and δ can be chosen such that

$$\|\mathbf{x}(0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}.$$

The origin is said to be *globally* asymptotically stable if it is asymptotically stable and solutions converge to the origin from all initial states in $\mathbb{R}^n$.

Nonautonomous systems

Consider the system (A.3), and suppose that $\mathbf{f}$ satisfies $\mathbf{f}(t, \mathbf{0}) = \mathbf{0} \quad \forall t \geq 0$, such that the origin is an equilibrium point of the system (A.3).

Definition A.4 ([78], Definition 4.4). The equilibrium point $\mathbf{x} = \mathbf{0}$ of (A.3) is

- uniformly stable (US) if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, independent of t_0 , such that

$$\|\mathbf{x}(t_0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| < \varepsilon, \quad \forall t \geq t_0 \geq 0,$$

- uniformly asymptotically stable (UAS) if it is US and there is a positive constant c , independent of t_0 , such that for all $\|\mathbf{x}(t_0)\| < c$, $\mathbf{x}(t) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$, uniformly in t_0 ; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that

$$\|\mathbf{x}(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta), \quad \forall \|\mathbf{x}(t_0)\| < c,$$

- uniformly globally asymptotically stable (UGAS) if it is US, $\delta(\varepsilon)$ can be chosen to satisfy $\lim_{\varepsilon \rightarrow \infty} \delta(\varepsilon) = \infty$, and, for each pair of positive numbers η and c , there is $T = T(\eta, c) > 0$ such that

$$\|\mathbf{x}(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta, c), \quad \forall \|\mathbf{x}(t_0)\| < c.$$

Equivalently, these properties can be characterized through class $\mathcal{K}$ and $\mathcal{KL}$ bounds on the solutions.

Lemma A.1 ([78], Lemma 4.5). *The equilibrium point $\mathbf{x} = \mathbf{0}$ of (A.3) is*

- *US if and only if there exist a class $\mathcal{K}$ function α and a positive constant c , independent of t_0 , such that*

$$\|\mathbf{x}(t)\| \leq \alpha(\|\mathbf{x}(t_0)\|), \quad \forall t \geq t_0 \geq 0, \quad \forall \|\mathbf{x}(t_0)\| < c,$$

- *UAS if and only if there exist a class $\mathcal{KL}$ function β and a positive constant c , independent of t_0 , such that*

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall \|\mathbf{x}(t_0)\| < c, \quad (\text{A.5})$$

- *UGAS if and only if inequality (A.5) is satisfied for any initial state $\mathbf{x}(t_0)$.*

Definition A.5 (Exponential stability, [78], Definition 4.5). The equilibrium point $\mathbf{x} = \mathbf{0}$ of (A.3) is exponentially stable if there exist positive constants c , k and λ such that

$$\|\mathbf{x}(t)\| \leq k\|\mathbf{x}(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \|\mathbf{x}(t_0)\| < c$$

and globally exponentially stable (GES) if the above inequality is satisfied for any initial state $\mathbf{x}(t_0)$.

Throughout this thesis, acronyms such as e.g. US may be used interchangeably to mean both "uniformly stable" and the property "uniform stability".

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